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VALUES OF NON-ATOMIC GAMES I:
THE AXIOMATIC APPROACH

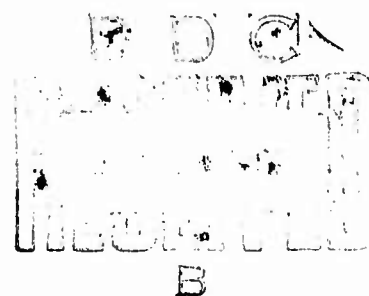
by

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INTRODUCTION

The value of an n -person game, n finite, was introduced in $[S_1]$; it is a function that associates to each player a number that, intuitively speaking, represents an a priori opinion of what it is worth to him to play in the game.

Games with a continuum of players have recently attracted considerable attention as models for mass phenomena in economics and game theory. * Here we will extend the definition of value to certain classes of games with a continuum of players, and investigate the properties of the concept so defined.

The paper is divided into several parts. In the first part, which appears herewith, we limit ourselves to an axiomatic approach, roughly parallel to the definition, via three axioms, of the value of a finite game as given in $[S_1]$. In subsequent parts, which we hope to publish later, we will investigate several "representation" theorems, roughly parallel to the approach to the value of a finite game via random orderings of the players, as presented toward the end of $[S_1]$.

Section 1 of this paper is devoted to various terminological and notational conventions. In Sec. 2 we describe the mathematical setting of our investigation, and in particular define the notions of

* Some of the papers using game models with a continuum of players are: $[Da, M-S, S_3]$ in the area of weighted majority games and their relatives; $[A_1, A_2, D_1, D_2, Hi, K_1, S_5, Sc_1, V]$ in the area of market games; $[K_2, P, S_2, Sc_2]$ in other areas.

"game" and "value". Section 3 is devoted to a statement of the main results of the paper, as well as some discussion and motivation. The remainder of the paper is devoted to proofs, as well as some examples and results of a technical nature. A guide to Secs. 4 through 10 will be found in Sec. 3.

This paper has been in the writing ever since the summer of 1963, and we have discussed it with many people since then. Among those with whom we have had valuable discussions are S. Agmon, S. Amitsur, A. Dvoretzky, T. E. Harris, Y. Kannai, Y. Lindenstrauss, B. Peleg, and B. Shitovitz. Particular thanks are due to Harry Furstenberg, who suggested the key idea of Sec. 8 (Lemma 8.6 and its proof).

1. SOME NOTATIONAL CONVENTIONS

The symbol $\| \quad \|$ for norm is used in many different senses throughout the paper; but it is never used in two different senses on the same space, so no confusion can result. In particular, when x is in a euclidean space of finite dimension (i. e., it is a finite-dimensional vector), then $\|x\|$ will always mean the maximum norm, i. e.,

$$\|x\| = \max_i |x_i|.$$

It is important to distinguish notationally between functions and their values. For example, if μ is a measure, then $\|\mu\|$ is its total variation whereas $\|\mu(S)\|$ is simply the absolute value of the real number $\mu(S)$.

Occasionally it will be necessary to use more than one norm on a given space; in that case the norms will be distinguished in various ways, for example, by subscript, as in $\| \quad \|_1$.

Closure will be denoted by a bar; thus \bar{A} is the closure of A .

Composition will usually be denoted by the symbol \circ ; thus if f is defined on the range of μ , then the function whose value at S is $f(\mu(S))$ will be denoted $f \circ \mu$. When no confusion can result, especially in the case of compositions of linear operators, the symbol \circ will occasionally be omitted.

The origin of any linear space (including, of course, the real line) will be denoted by 0; no confusion can result. In euclidean n -space, e^i will denote the i -th unit vector $(0, \dots, 0, 1, 0, \dots, 0)$, and e will denote the vector $(1, \dots, 1)$.

The symbol \subset will be used for inclusion that is not necessarily strict. Set-theoretic subtraction will be denoted by \setminus , whereas $-$ will be reserved for algebraic subtraction. $f|A$ will mean "f restricted to A".

2. DEFINITIONS OF GAME AND VALUE

In this section only, we will set off the basic definitions from the discussion by slightly indenting the former.

Let (I, \mathcal{C}) be a measurable space; that is, I is a set and \mathcal{C} is a σ -field of subsets of I . (I, \mathcal{C}) will be fixed throughout, and will be referred to as the underlying space. The term "set function" will always mean a real-valued function v on \mathcal{C} such that $v(\emptyset) = 0$.

In most of this paper (until Sec. 10) we will assume that

- (2.1) the underlying space (I, \mathcal{C}) is isomorphic* to $([0, 1], \mathcal{B})$, where $[0, 1]$ is the closed unit interval, and \mathcal{B} is the σ -field of Borel sets on $[0, 1]$.

This assumption is not as drastic as it seems; any Borel subset of any euclidean space, and indeed of any complete separable metric space, when considered as a measurable space, is isomorphic to $([0, 1], \mathcal{B})$ (see, e. g., [M]). We add that for much of the material of this paper (2.1) is not needed; we make the assumption because without it we would have to enter at a number of points into consideration of measure-theoretic pathologies that have nothing to do with

* Two measurable spaces are isomorphic if there is a one-one mapping from one onto the other that is measurable in both directions; the mapping is called an isomorphism.

the main subject of this paper. Section 10 is devoted to considering possible generalizations and pointing out exactly where (2.1) is used and in what way it might be modified.*

In the interpretation, a set function is a game, the members of I are players, and the members of \mathcal{C} are coalitions. The number $v(S)$, for $S \in \mathcal{C}$, is interpreted as the total payoff that the coalition S , if it forms, can obtain for its members; it will be called the worth of S . This way of representing a game is an obvious generalization of the standard representation in characteristic function form of a game with finitely many players $[N-M]$.

By the carrier of a game v we mean a coalition I' such that $v(S) = v(S \cap I')$ for all $S \in \mathcal{C}$. A coalition S is null if it is the complement of a carrier, and a player s is null if $\{s\}$ is null.** If all the players are null we say that the game is non-atomic; this agrees with the usual definition of "non-atomic" if v happens to be a measure. Though we will have little further technical use in this

* Another possible candidate for a "canonical" underlying space would have been $([0, 1], \mathcal{L})$ where \mathcal{L} is the σ -field of Lebesgue-measurable subsets of $[0, 1]$. We chose $([0, 1], \mathcal{B})$ instead, because the set functions that are non-atomic measures play a decisive role in our work, and the latter admits a much wider class of such measures. (Indeed, $([0, 1], \mathcal{B})$ admits non-atomic measures that are singular with respect to Lebesgue measure, whereas $([0, 1], \mathcal{L})$ admits only measures that have no singular component.)

** Null players are "dummies", in the sense of $[N-M]$, but dummies are not always null players, since games that differ by additive set functions may have different carriers.

paper for the notion of a general non-atomic game, most of the games to be considered here will in fact be non-atomic, and their non-atomicity is more or less the crux of the matter.

A set function v is said to be monotonic if $S \supset T$ implies $v(S) \geq v(T)$. The difference between two monotonic set functions is said to be of bounded variation. The set of all set functions of bounded variation forms a linear space over the field of real numbers, which will be called BV. Our investigations will focus on this space and certain of its subspaces.

An example of an element of BV is any set function of the form

$$v(S) = f(\mu(S))$$

where μ is a finite nonnegative measure and f is a function of bounded variation on the real interval $[0, \mu(I)]$ with $f(0) = 0$. Note that such games need not be non-atomic; indeed, they include all games with finite carriers (which are essentially the classical finite games of $[N-M]$, if we ignore the null players), as well as the generalized weighted-majority voting games, with denumerably or nondenumerably infinite sets of players. Value theories for these games have been investigated in $[S_1]$, $[S_4]$, $[M-S]$ and elsewhere, but our present preoccupation with non-atomic games gives a different cast to the problem, and no direct, formal connection is made.

The subspace of BV consisting of all bounded, finitely additive set functions, i. e., the bounded, finitely additive, signed measures on (I, \mathcal{C}) , will be denoted FA. Note that an element μ of FA is monotonic if and only if $\mu(S) \geq 0$ for all $S \in \mathcal{C}$.

Let Q be any subspace of BV. The set of monotonic games in Q will be denoted Q^+ . A mapping of Q into BV is called positive if it maps Q^+ into BV^+ , i. e., if it transforms monotonic set functions into monotonic set functions. It may of course happen that Q has no monotonic elements other than 0; in that case all linear mappings of Q into BV are positive.

Let \mathcal{H} denote the group of automorphisms of the underlying space (I, \mathcal{C}) , that is, isomorphisms of that space onto itself.* Each θ in \mathcal{H} induces a linear mapping θ_* of BV onto itself, defined by

$$(\theta_* v)(S) = v(\theta S).$$

A subspace Q of BV is called symmetric if $\theta_* Q = Q$ for all θ in \mathcal{H} .

We come at last to the definition of "value".

Let Q be a symmetric subspace of BV. A value on

* See the footnote on page 5.

Q is a positive linear mapping φ from Q into FA , such that for all Θ in \mathcal{K} and v in Q we have

$$(2.2) \quad \varphi \Theta_* = \Theta_* \varphi$$

and

$$(2.3) \quad (\varphi v)(I) = v(I).$$

We shall refer to φv as the value of the game v , and to $(\varphi v)(S)$ as the value of the coalition S .

Essentially, condition (2.2) says that the value does not depend on how the players are named, and (2.3) says that it distributes to the players the entire amount available to the all-player set. Similar conditions appear as Axioms 1 and 2 ("symmetry" and "efficiency") respectively in the original axiomatization $[S_1]$ of the value for finite games.*

* Axiom 2 in $[S_1]$ is apparently somewhat stronger than (2.3), in that it asserts $(\varphi v)(I') = v(I')$ for all carriers I' of v . However, for non-atomic games obeying (2.1) this follows from the conditions as given above. Indeed, in any game with infinitely many null players, we assert that all null coalitions have value zero. Together with (2.3), this is equivalent to the analogue of Axiom 2 in $[S_1]$.

To prove the assertion, deduce from (2.2) that all null players must get the same value, which must therefore vanish since the value is bounded. Hence all finite null coalitions get value 0. Now let S be an infinite null coalition, and let S_1 and S_2 be disjoint subcoalitions of S having the same cardinality as S . Whether they are denumerable or nondenumerable (and hence Borel sets with the power of the continuum), there is a symmetry of the game that takes S_1 onto S_2 ; hence they have the same value. It then follows easily that the value of S must be zero, as claimed.

If there are only finitely many null players, however, the argument fails. An example is the "unanimity" game (see Sec. 3) with a single

The linearity condition corresponds to Axiom 3 ("aggregation"). The positivity condition is not an axiom in $[S_1]$, but in that context (i. e., for finite games) it follows from the other axioms. Intuitively, in a monotonic game no coalition has negative worth itself, and it cannot decrease the worth of another coalition when joining it; the positivity condition accordingly requires that in such games all coalitions have nonnegative values.

Another reasonable condition that we might wish to impose on φ is continuity; but before that can be done, we must define a topology on BV . This will be done in Sec. 3. It will turn out that there is a close relationship between positivity and continuity, and that essentially nothing would be changed if we replaced the positivity condition in the definition of value by an appropriate continuity condition.

Still another likely condition, plausible but unneeded at present, is that of invariance under "strategic equivalence" ([N-M], page 245): if $v \in Q$ and $a \in FA \cap Q$, then $\varphi(v+a) = \varphi v + a$. With linearity, this amounts to saying that φ is a projection on FA , i. e., for any $v \in FA \cap Q$ we have $\varphi v = v$. (In the usual terminology, a game $v \in FA$ would be called "inessential".) Some discussion of the use of a projection axiom, in connection with the relaxation of assumption (2.1), will be found near the end of Sec. 10.

null player added. This game can be imbedded in a symmetric subspace of BV on which there is a unique value, which assigns value 1 to the null player in question.

A slightly different approach to the axiomatization of value is as follows: One speaks only of monotonic games v ; the value φv is always a nonnegative measure (finitely additive); and the linearity condition is replaced by the condition $\varphi(\alpha v + \beta w) = \alpha \varphi v + \beta \varphi w$, where α and β are nonnegative real numbers. Conditions (2.2) and (2.3) remain unchanged. This approach is entirely equivalent to the one we have adopted.

We mentioned above that the finite games may be considered members of BV , in the guise of set functions with finite carriers. It is easy to see that in the sense of the above definition there is a unique value on this subspace, which coincides with the value as originally defined in $[S_1]$. The situation as regards weighted-majority voting games with denumerably infinitely many players is less clear, since these games do not constitute a subspace. Presumably one can imbed them in a subspace of BV on which there is a unique value that coincides with that of $[S_4]$, but this has not been proved. Note, incidentally, that the space of all set functions with denumerable carriers is not a subspace of BV .

3. STATEMENT OF CHIEF RESULTS

In this section we will state and briefly motivate and discuss some of the main results of this paper. No proofs will be given in this section.

Unless otherwise specified, the word "measure" in this paper will refer to completely additive totally finite signed scalar measures. A "probability measure," as usual, is a nonnegative measure μ such that $\mu(I) = 1$. The subspace of BV consisting of all non-atomic measures on the underlying space (I, \mathcal{C}) will be denoted NA. In conformance with our notation, NA^+ will denote the cone of all nonnegative measures in NA. The space of all real-valued functions f of bounded variation on the unit interval $[0, 1]$, that obey $f(0) = 0$, will be denoted bv.

The first remark to be made about the definition of value is that there is no value on all of BV. Indeed, consider the "unanimity game" v defined by

$$v(S) = \begin{cases} 1 & \text{if } S = I \\ 0 & \text{otherwise;} \end{cases}$$

it is invariant under all automorphisms of the player space. Therefore if there is a value ϕ on BV, then ϕv must also be invariant under all automorphisms of the underlying space, by (2. 2); and no member

of FA can have this property unless $\phi v(I) = 0$, in which case (2.3) is violated.*

Although there is no value on all of BV, we will find that there are important subspaces of BV on which there is a value, which is moreover unique. In this paper we will characterize certain such subspaces, and investigate their properties and the properties of the values on them.

To state our results in their fullest generality, it will be useful to define a norm on BV. For v in BV define

$$\|v\| = \inf (u(I) + w(I)),$$

where the inf ranges over all monotonic set functions u and w such that

$$v = u - w.$$

The quantity $\|v\|$ will be called the variation** of v ; it is easily seen that it is a norm. Unless otherwise specified, all references in the sequel to topological notions (such as closure) on BV will be in the sense of the topology induced by the variation norm; in particular, if A is a subset of BV, then \bar{A} will always denote the closure of A in the variation norm. Similarly, the word "spanned" will always be used in the

* Note that the unanimity game is not non-atomic. However, we could define $v(S)$ to be 1 when S differs from I by at most a finite set, and 0 otherwise. This, too, is invariant under all automorphisms of the underlying space.

** A characterization of this norm that is more directly related to the intuitive notion of "variation" will be given in Sec. 4 (Proposition 4.1).

topological linear sense; i. e., the space spanned by a subset of BV is the closure of the set of all linear combinations of elements of that subset.

Set functions defined with the aid of non-atomic measures form the central subject of interest in this paper. Of particular importance are set functions of the form $f \cdot \mu$, where $f \in \text{bv}$ and μ is a non-atomic probability measure; they are called scalar measure games.^{*} The subspace of BV spanned by all scalar measure games will be denoted^{**} bvNA .

The question arises as to whether there is a value on bvNA . Unfortunately, the answer is no. Indeed, let μ be a non-atomic probability measure. Define f in bv by

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{if } 0 \leq x < 1, \end{cases}$$

and let $v = f \cdot \mu$. This may be called the " μ -almost unanimity game"; it is invariant under all automorphisms that preserve the coalitions of μ -measure 0. Therefore if there is a value ϕ on bvNA , then ϕv must also be invariant under all such automorphisms; and again no member of FA satisfying (2.3) has this property

^{*} I. e., games defined with the aid of a scalar measure, as distinguished from games defined with the aid of a vector measure.

^{**} More precise, but clumsier, would be " $(\text{bv})(\text{NA}^+)$ ". If we allowed $\mu \in \text{NA} \setminus \text{NA}^+$ we would introduce additional set functions, including some that are not even in BV; see Examples 5.7 and 5.9. The same remarks apply to $\text{bv}'\text{NA}$, defined next.

The difficulty in this example is caused by the discontinuity of f at 1; similar difficulties occur when there is a discontinuity at 0. Let us therefore define bv' to be the set of all bv functions that are continuous at 0 and at 1, and $bv'NA$ to be the subspace of BV spanned by all games of the form $f \cdot \mu$, where $f \in bv'$ and μ is a non-atomic probability measure.

THEOREM A. There is a unique value φ on $bv'NA$;
the range of φ is NA , and $\|\varphi\| = 1$. Furthermore, if
 $f \in bv'$ is such that $f(1) = 1$, and μ is a non-atomic
probability measure, then

$$\varphi(f \cdot \mu) = \mu.$$

The fact that $\|\varphi\| = 1$ implies in particular that $\|\varphi\|$ is finite, i.e., that φ is continuous. The relation between the positivity of φ and its continuity in the variation norm turns out to be basic to our investigation. This relation will be more closely examined in Sec. 4. In particular, if in the definition of value we replace the positivity condition by the condition that φ be continuous, then Theorem A remains true exactly as stated above.

The second sentence of Theorem A—the assertion that $\varphi(f \cdot \mu) = \mu$ —can be intuitively understood as follows: In the game $f \cdot \mu$, the payoff to a coalition depends only its μ -measure. There

* $\|\varphi\|$ is the norm of the operator φ , i.e., the supremum of $\|\varphi v\| / \|v\|$ over all nonzero v .

would therefore seem to be no reason to "discriminate" between coalitions having equal μ -measure, given assumption (2.1), and it seems natural to conjecture that they should get the same value. Because of the non-atomicity and the normalization condition (2.3), this implies that the value equals the μ -measure.

There is an analogy between the situation described in Theorem A and that in finite games that is worth pursuing. Recall that the "symmetry" axiom for finite games, on which (2.2) is modeled, may be replaced by an axiom that says that in symmetric games, i.e., games in which the payoff to a coalition depends only on the number of (nonnull) players in the coalition, the total value is divided equally among the nonnull players, i.e., the value is proportional to the number of players. Now in a finite game, the number of players in a coalition is a measure; moreover it is a measure which is in a certain sense "natural" or "distinguished"

In games with a continuum of players, there is nothing to distinguish one non-atomic probability measure from another (cf. Lemma 6.2). Therefore the continuous analogy of the finite situation would be that the value is proportional to the μ -measure whenever the payoff is a function of the μ -measure, for all μ -measures. This is precisely the content of the second sentence of Theorem A.

We remark that the difficult part of this theorem is the existence of the value; once this is known, the uniqueness and the

formula $\varphi(f \cdot \mu) = \mu$ follow rather easily. We will prove $\varphi(f \cdot \mu) = \mu$ in Sec. 6 on the assumption that there exists a value; whereas the proof of existence will be given only in Sec. 8.

Up to now we have dealt with set functions defined in terms of scalar measures, and with the space spanned by them. Let us now turn to "vector measure games"—i.e., set functions of the form $f \cdot \mu$, where $\mu = (\mu_1, \dots, \mu_n)$ is a non-atomic finite-dimensional vector measure and f is a real-valued function defined on the range of μ with $f(0) = 0$. We shall now see that under appropriate differentiability conditions on f , such set functions are already members of $bv'NA$.

To describe the situation we need a number of definitions. Let X be a convex subset of a euclidean space E^n . A vector z is said to be X-admissible if $z = x - y$ for some $x, y \in X$. Let f be a real function on X and let z be X-admissible. We shall say that f is continuously differentiable on X in the direction z if there is a real function on X which equals the derivative^{*} $df(x + \theta z)/d\theta$ at each point x in the relative interior of X , and which is continuous at each point in X (including the boundary). Such a function, if it exists, is unique; it will be denoted f_z , and will be called the derivative of f in the direction z .^{**} We shall say that f is continuously differentiable

^{*} Of course this involves the assumption that the derivative exists.

^{**} Note that f_z depends on the magnitude as well as the direction of z ; thus we have $f_{\alpha z} = \alpha f_z$ if z and αz are both X-admissible.

on X if for all X -admissible z , it is continuously differentiable on X in the direction z . Essentially, what the definition demands is that the directional derivatives of f exist and are continuous in the relative interior of X , and can be continuously extended to the boundary, for all those directions that do not "lead out" of the smallest linear manifold in which X lies. Of course, if X has full dimension, then these are simply all the directions. For example, if X is the closed unit cube in E^n , the definition simply says that each of the partial derivatives of f exists and is continuous on all of X , where $\partial/\partial x_j$ is defined in the one-sided sense when $x_j = 0$ or 1 .

THEOREM B. Let μ be a vector of measures in
NA, and let f be continuously differentiable on the range^{*}
of μ , with $f(0) = 0$; then $f \cdot \mu \in bv'NA$. Furthermore, if
 φ is the value on $bv'NA$, then

$$(3.1) \quad \varphi(f \cdot \mu)(S) = \int_0^1 f_{\mu(S)}(t\mu(I))dt,$$

where $f_{\mu(S)}$ is the derivative of f in the direction $\mu(S)$.

A restatement of formula (3.1) is of interest. Let R denote the range of μ . When R has full dimension, we have

$$(3.2) \quad \int_0^1 f_{\mu(S)}(t\mu(I))dt = \sum_{i=1}^n \mu_i(S) \int_0^1 f_i(t\mu(I))dt,$$

^{*} Recall Lyapunov's theorem [L], which says that the range of a non-atomic vector measure is convex and compact.

where $f_i = \partial f / \partial x_i$. More generally, let R have dimension $r \leq n$, and let z_1, \dots, z_r be a basis for the smallest linear subspace containing R . Then there are measures ν_1, \dots, ν_r such that

$$\mu(S) = \sum_{j=1}^r \nu_j(S) z_j,$$

so we obtain

$$\varphi(f \cdot \mu)(S) = \sum_{j=1}^r \nu_j(S) \int_0^1 f_{z_j}(t\mu(I)) dt.$$

Since each ν_j is a linear combination of the μ_i 's, we see that the value of a continuously differentiable vector measure game is always a linear combination of the component measures.

These formulas have the following startling aspect: they show that the value is completely determined by the behavior of f near the diagonal $[0, \mu(I)]$ (i. e., the set $\{t\mu(I): 0 \leq t \leq 1\}$); the behavior of f far from the diagonal is totally irrelevant. Intuitively, the reason is that for a coalition S chosen "at random" from \mathcal{C} , the ratios $\mu_i(S)/\mu_i(I)$, $i = 1, \dots, n$, tend to be equal, so that "almost all" of the points $\mu(S)$ fall on the diagonal. In fact, we do not even have to assume that f is differentiable far from the diagonal; we will actually prove (3.1) under the following condition, which is weaker than that stated in Theorem B:

(3.3) $f \cdot \mu$ in $bv'NA$ is such that μ is a vector of measures in NA and f is continuously differentiable on some convex^{*} neighborhood (in R) of the diagonal $[0, \mu(I)]$.

It is, however, impossible to dispense entirely with the differentiability condition. For example, consider the vector measure game defined by

$$(3.4) \quad v(S) = \max(\mu_1(S), \mu_2(S)),$$

where the underlying space is $[-1, 1]$ with its Borel subsets, λ is Lebesgue measure, and

$$\mu_1(S) = \lambda(S \cap [0, 1])$$

$$\mu_2(S) = \lambda(S \cap [-1, 0]);$$

then^{**} $v \notin bv'NA$.

The chief part of the proof of Theorem B will be given in Sec. 7, though the proof can only be completed after Theorem A is proved (in Sec. 8).

Theorem B can be strengthened in a number of other directions. First, the condition that f be continuously differentiable on R can

* We have defined "continuously differentiable" only for convex domains. Of course, any neighborhood of the diagonal contains a convex neighborhood of the diagonal.

** See Example 5.8, which is essentially the same game as (3.4).

be slightly relaxed on the boundary^{*} of R ; for details, see Sec. 9 (Proposition 9.17). Second, we can obtain information about the location of $f \cdot \mu$ in $bv'NA$. Indeed, let pNA denote the subspace of $bv'NA$ that is spanned by all powers of non-atomic probability measures. Then the conditions of Theorem B allow us to deduce that^{**} $f \cdot \mu \in pNA$.

That pNA is strictly contained in $bv'NA$ can be seen by means of the following theorem, with which we close this section.

THEOREM C. Let $v = f \cdot \mu$, where $\mu \in NA^+$ and f with $f(0) = 0$ is a real-valued function (not necessarily of bounded variation) on the range R of μ . Then $v \in pNA$ if and only if f is absolutely continuous on R .

Theorem C is proved in Sec. 5. In that section we also explore the effects of weakening the hypothesis from $\mu \in NA^+$ to $\mu \in NA$ (Proposition 5.5 and the following examples).

Additional results concerning the structure of and values on BV , $bv'NA$, pNA , and so on will be proved in the body of the paper.

Section 10 is devoted to generalizations, most of them in the direction of abandoning assumption (2.1).

^{*} (3.3) enables a far greater relaxation of the differentiability condition, but assumes a priori that $f \cdot \mu \in bv'NA$.

^{**} See Sec. 7.

4. BASIC PROPERTIES OF THE VARIATION NORM AND THE SPACE BV

A nondecreasing sequence of sets of the form

$$\emptyset = S_0 \subset S_1 \subset \dots \subset S_m = I$$

will be called a chain. A link of this chain is a pair of successive elements $\{S_{i-1}, S_i\}$. A subchain is a set of links; the chain itself may and will be identified with the subchain consisting of all the links.

If v is a set function and Λ is a subchain of a chain Ω , then the variation of v over Λ is defined by

$$\|v\|_{\Lambda} = \sum |v(S_i) - v(S_{i-1})|,$$

where the sum ranges over all indexes i such that $\{S_{i-1}, S_i\}$ is a link in the subchain Λ . For fixed Λ , the functional $\|v\|_{\Lambda}$ is a pseudonorm on BV, i.e., it enjoys all the properties of a norm except $\|v\| = 0 \Rightarrow v = 0$.

PROPOSITION 4.1. Let v be a set function. A necessary and sufficient condition that $v \in BV$ is that $\|v\|_{\Omega}$ be bounded over all chains Ω . If $v \in BV$, then

$$\|v\| = \sup \|v\|_{\Omega},$$

where the sup is taken over all chains Ω .

Remark. This characterization of BV and of the variation norm is basic throughout the entire paper. It will be used at least as often as the original definition; because of the frequent use, we will usually not refer explicitly to the proposition when using it.

Proof. Necessity: Let $v = u - w$, where u and w are monotonic. Then for any chain Ω

$$\|v\|_{\Omega} \leq \|u\|_{\Omega} + \|w\|_{\Omega} = u(I) + w(I).$$

This proves the necessity, and also that

$$\sup_{\Omega} \|v\|_{\Omega} \leq \|v\|.$$

Sufficiency: Assume that $\|v\|_{\Omega}$ is bounded. Define u to be the "upper variation" of v , i. e.,

$$u(S) = \sup \sum_i \max \{ (v(S_i) - v(S_{i-1})), 0 \},$$

where the sup is taken over all nondecreasing sequences

$\emptyset = S_0 \subset S_1 \subset \dots \subset S_k = S$. Let $w = u - v$. Then both u and w are monotonic. Indeed, if $T \supset S$, then

$$u(T) \geq \max \{ (v(T) - v(S)), 0 \} + u(S).$$

The right side of this inequality is $\geq u(S)$, proving that u is monotonic.

But it is also $\geq v(T) - v(S) + u(S)$, whence by transposing we obtain

$$w(T) = u(T) - v(T) \geq u(S) - v(S) = w(S),$$

proving that w is monotonic as well. This proves the sufficiency.

Finally, for a given $v \in BV$ let u and w be as in the sufficiency proof. For a given $\epsilon > 0$ let Ω be a chain

$$\emptyset = S_0 \subset S_1 \subset \dots \subset S_m = I$$

such that

$$\sum_i \max \{ (v(S_i) - v(S_{i-1})), 0 \} \geq u(I) - \epsilon.$$

Let Λ be the subchain consisting of all links such that $v(S_i) \geq v(S_{i-1})$, and Γ the subchain consisting of all the remaining links; then

$$\|v\|_{\Lambda} \geq u(I) - \epsilon.$$

Furthermore, we have

$$\|v\|_{\Lambda} - \|v\|_{\Gamma} = \sum_i (v(S_i) - v(S_{i-1})) = v(I),$$

and hence

$$\|v\|_{\Gamma} = \|v\|_{\Lambda} - v(I) \geq u(I) - v(I) - \epsilon = w(I) - \epsilon.$$

Hence from the definition of $\|v\|$ it follows that

$$\|v\|_{\Omega} = \|v\|_{\Lambda} + \|v\|_{\Gamma} \geq u(I) + w(I) - 2\epsilon \geq \|v\| - 2\epsilon.$$

Since ϵ is arbitrary, it follows that $\sup \|v\|_{\Omega} \geq \|v\|$. As the opposite inequality has already been proved, the proof of the proposition is complete.

COROLLARY 4.2. The inf in the definition of norm is achieved, i. e., there exist monotonic u and w with $v = u - w$ and $\|v\| = \|u\| + \|w\|$.

Proof. This follows immediately from the foregoing proof.

PROPOSITION 4.3. BV is complete, hence a Banach space.

Proof. Let $\{v_n\}$ be a Cauchy sequence. Then $\{v_n(S)\}$ is a Cauchy sequence for each $S \in \mathcal{C}$; denote its limit by $v(S)$. We must first show that v is of bounded variation. Let N be such that $\|v_n - v_N\| \leq 1$ whenever $n \geq N$. Then for each chain Ω and each $n \geq N$ we have

$$\|v_n\|_{\Omega} - \|v_N\| \leq \|v_n\|_{\Omega} - \|v_N\|_{\Omega} \leq \|v_n - v_N\|_{\Omega} \leq \|v_n - v_N\| \leq 1.$$

Letting $n \rightarrow \infty$, we deduce

$$\|v\|_{\Omega} \leq 1 + \|v_N\|;$$

hence v is of bounded variation. That $\|v_n - v\| \rightarrow 0$ is now easily verified, so the proposition is proved.

PROPOSITION 4.4. NA , FA , and the space of all countably additive members of FA are all closed subspaces of BV .

The straightforward proof is omitted.

PROPOSITION 4.5. For $v_1, v_2 \in BV$, we have

$$\|v_1 v_2\| = \|v_1\| \|v_2\|.$$

Proof. For monotonic u and w we have $\|uw\| = \|u\| \|w\|$.

From this and Corollary 4.2, it follows that for any v_1 and v_2 in BV ,

$$\begin{aligned} \|v_1 v_2\| &= \|(u_1 - w_1)(u_2 - w_2)\| \leq \|u_1 u_2\| + \|u_1 w_2\| + \|w_1 u_2\| + \|w_1 w_2\| \\ &= (\|u_1\| + \|w_1\|)(\|u_2\| + \|w_2\|) = \|v_1\| \|v_2\|. \end{aligned}$$

This completes the proof.

Proposition 4.5 shows that BV is a Banach algebra. In particular, it shows that multiplication is continuous, so that if X is a subalgebra of BV (linear subspace closed under multiplication), then the closure of X is also a subalgebra.

PROPOSITION 4.6. Let Q be a linear subspace of BV , and let φ be a linear operator from Q into M obeying the normalization condition (2.3) and having $\|\varphi\| \leq 1$.

Then φ is positive.

Proof. Let v be monotonic and suppose that contrary to the proposition, there is a coalition S with $(\varphi v)(S) < 0$. Then because of the monotonicity of v , $\|\varphi\| \leq 1$, and the normalization condition (2.3) we have

$$\begin{aligned} v(I) = \|v\| &\geq \| \varphi v \| \geq |(\varphi v)(S)| + |(\varphi v)(I) - (\varphi v)(S)| \\ &> |v(I) - (\varphi v)(S)| = v(I) + (\varphi v)(S) > v(I); \end{aligned}$$

this contradiction establishes the proposition.

We would now like to prove a converse to Proposition 4.6, i.e., a theorem that asserts that under appropriate conditions, the positivity of φ implies $\|\varphi\| \leq 1$. This, however, is not as straightforward as Proposition 4.6, and we must first introduce some additional concepts.

Let us call a subspace Q of BV reproducing^{*} if $Q = Q^+ - Q^+$. Reproducing subspaces of BV play a central role in this paper; in particular, we will show that pNA and $bv'NA$ are both reproducing. On reproducing subspaces Q of BV , we define a norm $\| \cdot \|_Q$, which is closely related to the variation norm $\| \cdot \|$, as follows:

$$\| v \|_Q = \inf (u(I) + w(I)),$$

^{*}The term "reproducing" has been used in [Kr], in a sense closely related but not quite identical to the current one.

where the inf ranges over all monotonic set function u and w in Q such that

$$v = u - w.$$

The reader will note that this definition differs from that of the variation norm only in that u and w are now required to be in Q .

Clearly

$$(4.7) \quad \|v\| \leq \|v\|_Q.$$

The norm $\|\cdot\|_Q$ will be called the internal norm on Q or the Q -norm.

Unless otherwise specified, topological terms (such as "closed") will continue to refer to the variation norm.

PROPOSITION 4.8. Let Q be a closed subspace of BV , and let $B = Q^+ - Q^+$. Then B is reproducing, and is complete in the B -norm.

Proof. We have

$$Q^+ \subset B \cap BV^+ = B^+ = (Q^+ - Q^+)^+ \subset Q^+,$$

whence $Q^+ = B^+$ and $B = Q^+ - Q^+ = B^+ - B^+$; hence B is reproducing.

The completeness is somewhat less immediate; we first prove two lemmas.

LEMMA 4.9. Q^+ is closed.

Proof. Let $v_i \in Q^+$, $v_i \rightarrow v$. Since Q is closed, we have $v \in Q$. Since $v_i \rightarrow v$ in the norm of BV , it follows that $v_i(S) \rightarrow v(S)$ for each S . Hence if $S \supset T$ we have

$$v(S) - v(T) = \lim (v_i(S) - v_i(T)) \geq 0,$$

and the proof is complete.

LEMMA 4.10. For every $v \in B$ there is a $u \in Q^+$
such that $u - v$ is monotonic and

$$\|u\|_B \leq 2\|v\|_B.$$

Proof. If $v = 0$ we may take $u = v$. If $v \neq 0$ then for each $\epsilon > 0$ we can find u and w in Q^+ such that

$$u(I) + w(I) \leq \|v\|_B + \epsilon;$$

if we set $\epsilon = \|v\|_B$ and note that $w(I) \geq 0$, then the lemma follows.

Proof of Proposition 4.8. Let $\{v_i\}$ be a Cauchy sequence in B in the internal norm $\|\cdot\|_B$. From (4.7) it follows that it is also a Cauchy sequence in the variation norm $\|\cdot\|$, and hence has a limit v in BV , which must be in Q since Q is closed. Now we may find a subsequence $\{v_{i_j}\}$ of $\{v_i\}$ such that

$$(4.11) \quad \|v_m - v_{i_j}\|_B \leq \frac{1}{2^j}$$

for all $m \geq i_j$. Let u_1 correspond to v_{i_1} in accordance with Lemma 4.10, and, for $j > 1$, let u_j correspond to $v_{i_j} - v_{i_{j-1}}$. Let

$$\hat{u}_j = u_1 + \dots + u_j.$$

Since $u_1 - v_{i_1}$, $u_2 - (v_{i_2} - v_{i_1})$, \dots , $u_j - (v_{i_j} - v_{i_{j-1}})$ are all monotonic, it follows that their sum $\hat{u}_j - v_{i_j}$ is also monotonic, hence in Q^+ . Now by (4.11) and by Lemma 4.10, $\{\hat{u}_j\}$ is a Cauchy sequence in the B-norm, hence a fortiori in the variation norm.

Hence it converges to a limit u in the variation norm, and by Lemma 4.9 we have $u \in Q^+$. Hence $\hat{u}_j - v_{i_j} \rightarrow u - v$ in the variation norm. Again applying Lemma 4.9, we deduce that $u - v \in Q^+$. Hence

$$v = u - (u - v) \in Q^+ - Q^+ = B.$$

It remains to prove that $v_i \rightarrow v$ in the B-norm. By (4.11) it is sufficient to show that $\|v - v_{i_j}\|_B \rightarrow 0$. Now $u - \hat{u}_j$ is the limit of $u_{j+1} + \dots + u_k$ as $k \rightarrow \infty$, hence by Lemma 4.9 it is in Q^+ . Similarly $u - \hat{u}_j - (v - v_{i_j})$ is the limit of

$$[u_{j+1} - (v_{i_{j+1}} - v_{i_j})] + \dots + [u_k - (v_{i_k} - v_{i_{k-1}})]$$

as $k \rightarrow \infty$, and hence it too is in Q^+ . Hence

$$\|v - v_{i_j}\|_B \leq (u(I) - \hat{u}_j(I)) + (u(I) - \hat{u}_j(I) - (v(I) - v_{i_j}(I))) \rightarrow 0,$$

since $\|u - \hat{u}_j\| \rightarrow 0$ and $\|v - v_{i_j}\| \rightarrow 0$. This completes the proof of Proposition 4.8.

We are now ready for the converse to Proposition 4.6 to which we referred above.

PROPOSITION 4.12. Let A be a reproducing linear
subspace of BV such that

$$\|v\| = \|v\|_A$$

for all $v \in A$. Let $Q = \bar{A}$. Then Q is reproducing, and

$$\|v\| = \|v\|_Q$$

for all $v \in Q$. Furthermore, if φ is a positive linear
operator from Q into FA, then

$$\|\varphi\| \leq 1.$$

Proof. Let $B = Q^+ - Q^+$; by Proposition 4.8 B is reproducing
and is complete in the B-norm, and $A \subset B$. Now let A^* be the
closure of A, as a subspace of B, in the B-norm; we have

$$A \subset A^* \subset B \subset \bar{B} = Q \subset BV.$$

Then we claim that

$$(4.13) \quad \|v\| = \|v\|_B$$

for all $v \in A^*$. Indeed, when $v \in A$ then (4.13) follows from

$$\|v\| \leq \|v\|_B \leq \|v\|_A = \|v\|.$$

When $v \in A^* \setminus A$, let $\{v_i\}$ be a sequence in A such that $v_i \rightarrow v$ in
the B-norm. Then

$$\|v_i\| = \|v_i\|_B,$$

$$|\|v_i\|_B - \|v\|_B| \leq \|v_i - v\|_B \rightarrow 0,$$

and

$$|\|v_i\| - \|v\|| \leq \|v_i - v\| \leq \|v_i - v\|_B \rightarrow 0;$$

hence

$$\|v\| = \lim \|v_i\| = \lim \|v_i\|_B = \|v\|_B,$$

proving (4.13).

Now A^* is a closed subspace of B in the B -norm, and since B is complete in this norm, it follows that A^* is also complete in the B -norm. But then from (4.13) it follows that A^* is complete in the variation norm as well, and hence it is a closed subspace of BV . Thus

$$A^* = \overline{A^*} \supset \bar{A} = Q \supset B \supset A^*;$$

hence equality holds throughout, and in particular $Q = B = Q^+ - Q^+$.

This proves that Q is reproducing. Hence the Q -norm is defined on Q , and from (4.13) and $A^* = B = Q$ we obtain

$$\|v\| = \|v\|_Q$$

for all $v \in Q$.

Now let φ be a positive linear operator from Q into FA . Let $v \in Q$, and for given $\epsilon > 0$ let u and w in Q^+ be such that

$$v = u - w$$

and

$$\|v\|_Q + \epsilon \geq u(I) + w(I).$$

Then

$$\begin{aligned} (4.14) \quad \|\varphi v\| &= \|\varphi u - \varphi w\| \leq \|\varphi u\| + \|\varphi w\| \\ &= (\varphi u)(I) + (\varphi w)(I) = u(I) + w(I) \leq \|v\|_Q + \epsilon = \|v\| + \epsilon, \end{aligned}$$

and $\|\varphi\| \leq 1$ follows easily. This completes the proof of

Proposition 4.12.

The remainder of this section will not be heavily used in the sequel; but it is of some interest in its own right.

PROPOSITION 4.15. Let Q be a closed reproducing subspace of BV ; then the variation norm and the internal norm on Q are equivalent, i.e., for some γ ,

$$\|v\|_A \leq \gamma \|v\|$$

for all $v \in A$. Furthermore, if φ is a positive linear operator from Q into FA , then φ is continuous.

Remark. The first sentence of this proposition is a kind of converse to that part of Proposition 4.12 that says that if $\|\cdot\| = \|\cdot\|_A$, then $Q = \bar{A}$ is reproducing.

Proof. The proof follows immediately from the following theorem of Bakhtin, Krasnoselskii, and Stetzenko [R-Kr-S]^{*}: Let E_1 and E_2 be two Banach spaces, and let K_1 and K_2 be cones in E_1 and E_2 respectively (i. e., if $x, y \in K_1$ and α, β are nonnegative real numbers, then $\alpha x + \beta y \in K_1$). Assume

- (i) K_1 and K_2 are closed;
- (ii) $E_1 = K_1 - K_1$;
- (iii) $x, y \in K_2$ implies $\|x\| \leq \|x + y\|$.

Let ψ be an operator from E_1 to E_2 such that $\psi K_1 \subset K_2$.

Then ψ is continuous.

In our case we may take E_1 to be Q with the variation norm, E_2 to be Q with the internal norm, ψ to be the identity, and $K_1 = K_2 = Q^+$. The conclusion that the identity is continuous is precisely the conclusion that $\|v\|_A \leq \gamma \|v\|$. The conditions are readily verified; the only ones worthy of special notice are condition (ii) and the condition that E_2 be a Banach space. Condition (ii) says precisely that Q is reproducing. That E_2 is a Banach space follows from Proposition 4.8 above. Proposition 4.8 is, however, not necessary for the proof of the current proposition; for, if Q were not complete in the internal norm we could, for the proof of this proposition, substitute its completion.

^{*} See also [Kr], the footnote on p. 64.

To prove the last sentence of our proposition, we can imitate (4.14), using $\|v\|_Q \leq \gamma \|v\|$, and deduce that $\|\varphi\| \leq \gamma$. However, it is not necessary to use the notion of internal norm at all; instead, one can use the [B-Kr-S] theorem directly. Indeed, take $E_1 = Q$, $E_2 = FA$ (both with the variation norm), $K_1 = E_1^+$, and $\psi = \varphi$. Condition (ii) says that Q is reproducing, condition (iii) says that $0 \leq v \leq w$ implies $\|v\| \leq \|w\|$, and $\psi K_1 \subset K_2$ says that φ is positive. The conclusion that ψ is continuous is then precisely what is needed.

This completes the proof of Proposition 4.15.

In connection with values, the chief consequence of Proposition 4.15 is that on reproducing spaces, every value is continuous. This is interesting because the positivity condition in the definition of value has content only to the extent that Q is reproducing. Thus the positivity condition has no direct bearing on set functions v in Q that are not the difference of monotonic functions; and if Q has no nontrivial reproducing subspaces—i.e., $Q^+ = \{0\}$ —then the positivity condition is vacuous. Thus Proposition 4.15 can be interpreted to mean that whenever the positivity condition is fully effective, then it implies continuity.

Continuity of the value is the chief tool in proving a number of uniqueness theorems, for example the uniqueness of the value on pNA (or bv'NA); the importance of knowing that a space is

reproducing is therefore evident. But the condition is an elusive one, because often we can establish it only via Proposition 4.12—i. e., by first proving that $\| \cdot \| = \| \cdot \|_A$; and once we know the latter, we no longer need the reproducingness and can proceed directly (cf. the end of Sec. 7). Under certain conditions one can show directly (i. e., without considering internal norms) that a space is in a sense "almost reproducing"; this will be discussed in Appendix B. Unfortunately, we have not succeeded in deducing from this notion of "almost reproducing" that the value must be continuous.

A good deal of this section has been devoted to finding conditions under which the value is continuous—in the variation norm. In Sec. 3, we saw that on the spaces that are of particular interest in this paper, such as $bv'NA$, the value is indeed continuous—again in the variation norm. It is therefore of some interest to note that the variation norm is crucial here; that, for example, the value is not continuous in the supremum norm: $\|v\|' = \sup_S |v(S)|$. Indeed, let $\{g_n(x)\}$ be a sequence of polynomials such that $g_n'(0) = 1$ for all n , and such that $g_n(x) \rightarrow 0$ as $n \rightarrow \infty$, uniformly for $|x| \leq 1$. For example, take $g_n(x) = x(1-x^2)^n$. Suppose that $I = [-1, 1]$, that

$$\mu_1(S) = \lambda(S \cap [-1, 0])$$

$$\mu_2(S) = \lambda(S \cap [0, 1]),$$

and that

$$v_n(S) = g_n(\mu_1(S) - \mu_2(S)).$$

Then $\|v_n\|' = \max_{|x| \leq 1} |g_n(x)| \rightarrow 0$ as $n \rightarrow \infty$; but from formula (3.1) applied to $f(x_1, x_2) = g_n(x_1 - x_2)$ we deduce $\varphi(v_n) = \mu_1 - \mu_2$, and hence

$$\|\varphi(v_n)\|' = \sup |\mu_1(S) - \mu_2(S)| = 1$$

for all n , so $\|\varphi(v_n)\|' \neq 0$.

It might be thought that the above phenomenon is a consequence of our indirectly building the variation into our value notion, via the positivity condition. But this is not the case. Indeed, on the space of all linear combinations of powers of measures in NA^+ —of which the above v_n are members (see Lemma 7.2)—the value is uniquely determined even without the positivity condition.^{*} Thus, at least for this rather restricted space, considerations of positivity and the variation norm arise naturally out of the other conditions for the value.

^{*}The value on this space is determined by Proposition 6.1, whose proof makes no use of the positivity condition.

5. THEOREM C.

We start this section with a number of concepts and propositions that will be needed in the proof of Theorem C, and also subsequently. (The statement of Theorem C will be found in Sec. 3.)

If v and w are set functions, then v is said to be absolutely continuous with respect to w (written $v \ll w$) if for every $\epsilon > 0$ there is a $\delta > 0$ such that for every chain Ω and every subchain Λ of Ω ,

$$(5.1) \quad \|w\|_{\Lambda} \leq \delta \Rightarrow \|v\|_{\Lambda} \leq \epsilon.$$

Note that the relation \ll is transitive, and that if v and w are measures, it coincides with the usual notion of absolute continuity.

A set function v is said to be absolutely continuous if there is a measure $\mu \in NA^+$ such that $v \ll \mu$. The set of all absolutely continuous set functions in BV is denoted AC .

PROPOSITION 5.2. AC is a closed subspace of BV .

Proof. AC is easily seen to be a linear space. To prove $AC \subset BV$, let $v \ll \mu$, $\mu \in NA^+$, and let δ correspond to $\epsilon = 1$ in accordance with (5.1). Given a chain Ω , we claim that

$$\|v\|_{\Omega} \leq (2\mu(I)/\delta) + 1. \text{ Indeed, because of the non-atomicity of } \mu$$

we may assume w. l. o. g. that

$$|\mu(S_i) - \mu(S_{i-1})| < \frac{1}{2} \delta$$

for all i . We may then partition Ω into subchains Λ , for example by taking sets of consecutive links, such that for all but the last subchain, we have

$$\frac{1}{2} \delta \leq \|\mu\|_{\Lambda} < \delta,$$

and for the last subchain we have

$$0 < \|\mu\|_{\Lambda} < \delta.$$

Since the variation of v over each such subchain is at most 1, and the number of such subchains is at most one more than $2\mu(I)/\delta$, our claim is proved. Hence $v \in BV$.

It remains to show that AC is closed as a subspace of PV .

Let $\|v_i - v\| \rightarrow 0$, where $v_i \ll \mu_i$ and $\mu_i \in NA^+$. W.l.o.g. assume $\mu_i(I) = 1$, all i , and set

$$u = \sum_{i=1}^{\infty} \left(\frac{1}{2}\right)^i \mu_i.$$

Then $\mu_i \ll u$ for all i , and hence $v_i \ll u$ for all i . Now for given ϵ , let v_i be such that $\|v_i - v\| < \frac{1}{2}\epsilon$, and let δ be such that for any subchain Λ ,

$$\|\mu\|_{\Lambda} < \delta \Rightarrow \|v_i\|_{\Lambda} < \frac{1}{2}\epsilon.$$

Then $\|\mu\|_{\Lambda} < \delta$ implies

$$\|v\|_{\Lambda} \leq \|v_i\|_{\Lambda} + \|v - v_i\|_{\Lambda} < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon.$$

Thus $v \ll u$, and so $v \in AC$. This completes the proof of the proposition.

COROLLARY 5.3. $pNA \subset AC$.

Proof. Clearly AC contains all powers of non-atomic measures; the corollary then follows from Proposition 5.2.

The next lemma is a simple consequence of Lyapunov's theorem [L].

LEMMA 5.4. Let $\mu = (\mu_1, \dots, \mu_n)$ be a vector of measures in NA, and let S^1 and S^0 in \mathcal{C} be such that $S^1 \supset S^0$. Then we may construct a family of sets $\{S^\alpha : 0 \leq \alpha \leq 1\} \subset \mathcal{C}$, in such a way that

$$\mu(S^\alpha) = \alpha \mu(S^1) + (1 - \alpha) \mu(S^0),$$

and that $\alpha > \beta$ implies $S^\alpha \supset S^\beta$.

Proof. Apply Lyapunov's theorem to the measure space consisting of $S^1 \setminus S^0$ and its measurable subsets to obtain a set $S^{\frac{1}{2}}$ such that $S^1 \supset S^{\frac{1}{2}} \supset S^0$ and

$$\mu(S^{\frac{1}{2}}) = \frac{1}{2} \mu(S^1) + \frac{1}{2} \mu(S^0).$$

Then apply the theorem again to $S^1 \setminus S^{\frac{1}{2}}$ and $S^{\frac{1}{2}} \setminus S^0$. Continuing in this way, we may define S^α for each dyadic rational α , in a way that satisfies the conditions of the lemma. Now if β is an arbitrary member of $[0, 1]$, define

$$S^\beta = \bigcup_{\alpha \leq \beta} S^\alpha,$$

the union being extended over all α that are dyadic rationals. This completes the proof of the lemma.

We are now ready for the

Proof of Theorem C (see page 21). First suppose f is absolutely continuous. Then $f(x) = \int_0^x g(t)dt$ for all x in the range R of μ , where $g \in L^1 = L^1(R)$. It is well known that g can be approximated in L^1 by polynomials; this follows from the fact that it can be approximated by step functions, which can be approximated by continuous functions, which can be approximated by polynomials. If q is a polynomial with $\int_R |q(t) - g(t)|dt < \epsilon$, and if $p(x) = \int_0^x q(t)dt$, then the variation of $p - f$ is $< \epsilon$. Hence $f(x)$ can be approximated in variation by polynomials in x , and so from the nonnegativity of μ it follows that v can be approximated in variation by the corresponding polynomials in μ . Hence $v \in \text{pNA}$.

To complete the proof, let $v = f \cdot \mu \in \text{pNA}$. By Corollary 5.3, $f \cdot \mu \in \text{AC}$, so there is a $\nu \in \text{NA}^+$ such that $f \cdot \mu \ll \nu$. W.l.o.g. we may assume that both μ and ν are probability measures. Applying Lemma 5.4 to the vector measure (μ, ν) , we may assign to each $\alpha \in [0, 1]$ a set S^α in such a way that $S^0 = \emptyset$, $S^1 = I$, $\alpha > \beta$ implies $S^\alpha \supset S^\beta$, and $\mu(S^\alpha) = \nu(S^\alpha) = \alpha$. If we now apply the definition of $f \cdot \mu \ll \nu$ to chains of sets of the form S^α , we deduce that for any finite union of disjoint intervals in $[0, 1]$, the sum of the variations of f over these intervals tends to 0 as their total length tends to 0; and this is precisely one of the definitions of absolute continuity of f . This completes the proof of Theorem C.

If $\mu \in \text{NA}$ is permitted to take negative as well as positive values, the forward implication in Theorem C remains true, i.e., $f \cdot \mu \in \text{pNA}$ implies the absolute continuity of f . Indeed, let $I = I^+ \cup I^-$ be a Hahn decomposition of I , that is, μ is nonnegative on subsets of I^+ and non-positive on subsets of I^- [H_1 , p. 121]. Clearly the range R of μ is $[\mu(I^-), \mu(I^+)]$. Now apply the theorem to the player spaces I^+ and I^- separately; it follows that f is absolutely continuous on $[\mu(I^-), 0]$ and on $[0, \mu(I^+)]$, and hence on all of R .

However, the converse is no longer true if μ is permitted to be in $\text{NA} \setminus \text{NA}^+$. In fact, absolutely continuous functions f exist for which $f \cdot \mu \notin \text{BV}$, and, a fortiori, $f \cdot \mu \notin \text{pNA}$; they also exist for which $f \cdot \mu$ is in BV , but is still not in pNA . The next proposition and the three examples that follow will help to explain the situation.

If R is any real interval containing the origin, we let $\text{bv}(R)$ denote the space of real-valued functions f of bounded variation that obey $f(0) = 0$; thus, we have $\text{bv} = \text{bv}([0, 1])$.

PROPOSITION 5.5. Let $v = f \cdot \mu$, where $\mu \in \text{NA}$
has range $R = [-a, b]$, with $-a < 0 < b$. Let

$$g(x, y) = (x+a)(b-y) \frac{f(y) - f(x)}{y - x}.$$

Then $v \in \text{BV}$ if and only if $f \in \text{bv}(R)$ and $|g(x, y)|$
is bounded in the domain $-a < x < y < b$.

Remark. If f happens to be differentiable, then the condition

on g says that the derivative of f must be bounded, except that near the boundary of R it is permitted to grow as fast as the reciprocal of the distance to the boundary. It is interesting that the conditions on f do not involve μ , except through its range R .

The underlying idea of the proof which follows is quite simple. Intuitively, we may think of the variation of f as being accumulated by a moving point, x , as it sweeps once across the domain R . For the variation of $f \circ \mu$, however, we must allow x to sweep back and forth within R , but with the proviso that the total distance traveled to the right (resp. left) must not exceed b (resp. a). Thus, if f becomes arbitrarily steep at some point in the interior of R (as in Example 5.7 below), then we can construct chains that oscillate in the neighborhood of that point and thereby accumulate an arbitrarily high variation for $f \circ \mu$. But if the steepness occurs only at the endpoints of R , then the moving point may not be able to remain in the neighborhood long enough to do any damage.

Proof of Proposition 5.5. First assume $f \circ \mu \in BV$. Given any increasing sequence $0 < x_1 < \dots < x_p = b$, we can construct a chain (cf. Lemma 5.4) whose first p elements (after \emptyset) satisfy $\mu(S_i) = x_i$. This shows that $f \in bv([0, b])$. Similarly, $f \in bv([-a, 0])$; hence $f \in bv(R)$ as required.

To show that $g(x, y)$ is bounded, let k be a positive integer, and split $[x, y]$ into k equal intervals. At least one of them will

have endpoints x', y' satisfying

$$|f(y') - f(x')| \geq |f(y) - f(x)|/k,$$

as well as

$$x \leq x' < y' \leq y \text{ and } y' - x' = (y - x)/k.$$

if k is large enough, there will be a positive integer m such that

$$(y' - x')m \leq \min(a, b, x'+a, b-x').$$

Using Lyapunov's theorem (drawing the increments $S_i \setminus S_{i-1}$ for $i > 1$

alternately from the positive and negative sides of a Hahn

decomposition of I), we can construct a chain

$\Omega: \emptyset = S_0 \subset S_1 \subset \dots \subset S_{2m+1} \subset I$ with the property

$$\mu(S_1) = \mu(S_3) = \dots = \mu(S_{2m+1}) = x'$$

$$\mu(S_2) = \mu(S_4) = \dots = \mu(S_{2m}) = y'.$$

(The conditions on m ensure that we have enough room for this much maneuvering.) Then

$$\begin{aligned} \|f \circ \mu\|_{\Omega} &= |f(x')| + 2m |f(y') - f(x')| + |f(x') - f(\mu(I))| \\ &\geq \frac{2m}{k} |f(y) - f(x)| = \frac{2m(y-x) |g(x,y)|}{k(x+a)(b-y)}. \end{aligned}$$

Suppose now that m was chosen as large as possible. Then we have

$$\begin{aligned}
 m + 1 &> \frac{1}{y'-x'} \min(a, b, x'+a, b-x') \\
 &\geq \frac{k}{y-x} \min(a, b, \frac{(x'+a)(b-x')}{b+a}) \\
 &= \frac{k(x'+a)(b-x')}{y-x} \min(\frac{\min(a, b)}{(x'+a)(b-x')}, \frac{1}{b+a}) \\
 &\geq \frac{k(x+a)(b-y)C}{y-x}
 \end{aligned}$$

where $C > 0$ is independent of x , y , and k . Hence

$$\|f \cdot \mu\|_{\Omega} \geq \frac{2mC}{m+1} |g(x, y)| \geq C |g(x, y)|.$$

Since $\|f \cdot \mu\|_{\Omega} \leq \|v\|$, this shows that g is bounded. This completes the proof in one direction.

For the other direction, let V denote the total variation of f on R ; let G denote the supremum of $|g(x, y)|$ on $-a < x < y < b$; and let Ω be an arbitrary chain. We must show that $\|v\|_{\Omega}$ is bounded.

Let X denote the set of values assumed by $\mu(S)$ for $S \in \Omega$.

First we wish to make precise the idea that the set X cannot get close to both endpoints of R . Let b' be the largest number in X and $-a'$ the smallest. Then we assert

$$(5.6) \quad \max(b-b', a-a') \geq \min(a/2, b/2).$$

To prove this, let $S_i, S_j \in \Omega$ be such that $\mu(S_i) = b'$ and $\mu(S_j) = -a'$, and suppose $i < j$. Let Ω_0 be the chain $\{\emptyset, S_i, S_j, I\}$. Then since $-a' \leq \mu(I) = b - a$, we have

$$\|\mu\|_{\Omega_0} = b' + |-a' - b'| + |b - a + a'| = 2b' + 2a' - a + b.$$

But $\|\mu\|_{\cap_0} \leq \|\mu\| = a + b$; hence $a - a' \geq b'$ and we have

$$\max(b - b', a - a') \geq \max(b - b', b') \geq b/2.$$

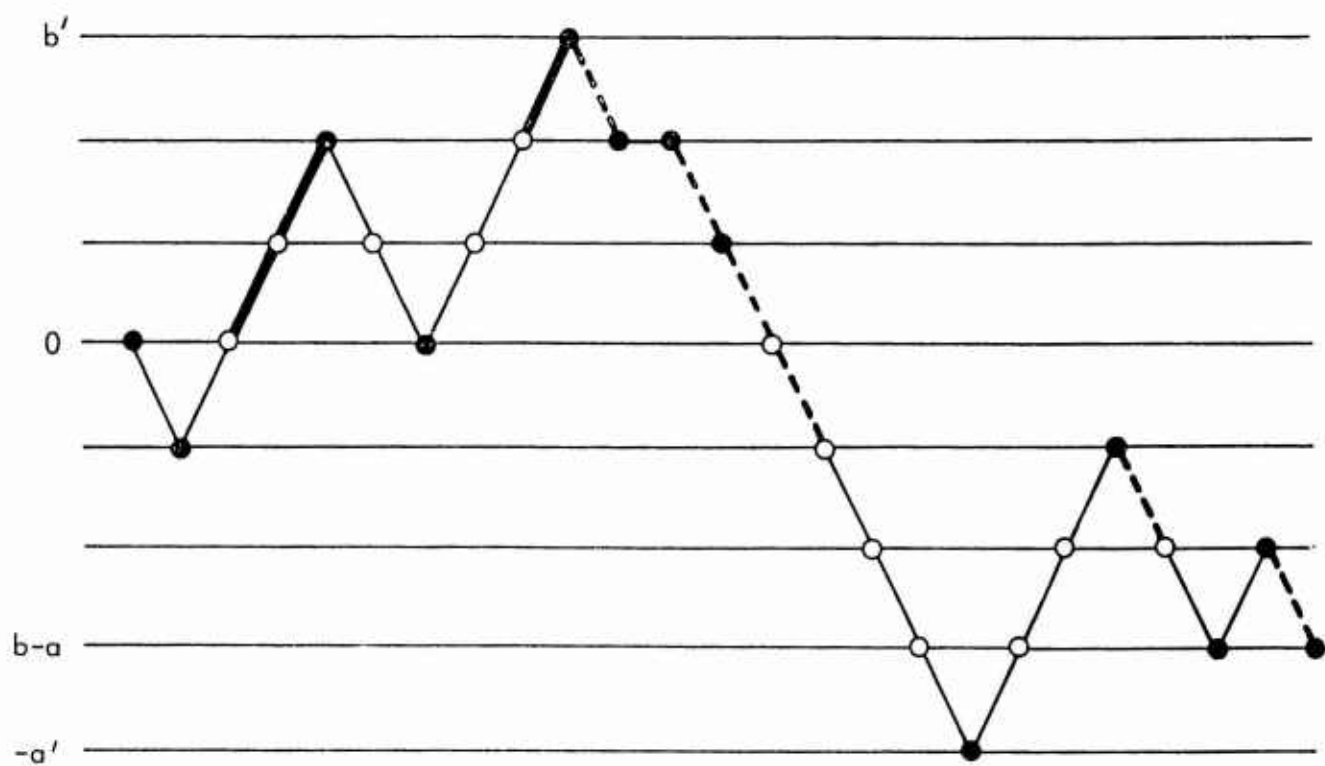
If $j < i$, a similar argument gives the estimate $a/2$ instead. This completes the proof of (5.6).

Next, we define an auxiliary chain Ω^* as follows: For each link (S, T) of Ω for which the set $Y = X \cap (\mu(S), \mu(T))$ is not empty, we introduce intermediate sets using Lyapunov's theorem, in such a way that each of the values in Y is assumed in its natural order (i. e., in increasing sequence if $\mu(S) < \mu(T)$, decreasing if $\mu(S) > \mu(T)$). The resulting, enlarged chain, denoted by Ω^* , has the property that the sequence of values $\mu(S_i^*)$ does not skip over any value that is assumed elsewhere in the sequence. By the triangle inequality, we have $\|f \circ \mu\|_{\cap^*} \geq \|f \circ \mu\|_{\cap}$.

We now define two subchains of Ω^* (see Fig. 1). Let Λ_1 be the set of links $\{S_{i-1}^*, S_i^*\}$ such that $\mu(S_i^*) > 0$ and $\mu(S_j^*) \neq \mu(S_i^*)$ for all $j < i$. Let Λ_2 be the set of links $\{S_{i-1}^*, S_i^*\}$ such that $\mu(S_{i-1}^*) > b - a$ and $\mu(S_j^*) \neq \mu(S_{i-1}^*)$ for all $j > i-1$. Clearly $\Lambda_1 \cap \Lambda_2 = \emptyset$, and we have $\|\mu\|_{\Lambda_1} = b'$ and $\|\mu\|_{\Lambda_2} = b' - (b - a)$. Since $\|\mu\|_{\cap^*} \leq \|\mu\| = a + b$, we have

$$\|\mu\|_{\cap^* \setminus \Lambda_1 \setminus \Lambda_2} \leq a + b - b' - (b' - (b - a)) = 2(b - b').$$

Hence



Key: Ω \odot
 Ω^* \circ and \otimes
 Λ_1 ———
 Λ_2 - - - -

Figure 1

$$\begin{aligned} \|f \cdot \mu\|_{\Omega^* \setminus \Lambda_1 \setminus \Lambda_2} &\leq \max_i \left| \frac{f(\mu(S_i^*)) - f(\mu(S_{i-1}^*))}{\mu(S_i^*) - \mu(S_{i-1}^*)} \right| \cdot \|\mu\|_{\Omega^* \setminus \Lambda_1 \setminus \Lambda_2} \\ &\leq 2(b-b') \max_{\substack{x, y \in X \\ x < y}} \frac{|f(y) - f(x)|}{y - x}. \end{aligned}$$

On the other hand, $\|f \cdot \mu\|_{\Lambda_1}$ and $\|f \cdot \mu\|_{\Lambda_2}$ are each bounded by V , the variation of f . Hence

$$\|f \cdot \mu\|_{\Omega^*} \leq 2V + 2(b-b') \max_{\substack{x, y \in X \\ x < y}} \frac{|f(y) - f(x)|}{y - x}.$$

If $b = b'$ we therefore have $\|f \cdot \mu\|_{\Omega^*} \leq 2V$; the same holds if $a = a'$, by a symmetric argument that begins by redefining Λ_1 by $\mu(S_i^*) < 0$ instead of $\mu(S_i^*) > 0$. So we may continue under the assumption that both $b > b'$ and $a > a'$. Then

$$\begin{aligned} \|f \cdot \mu\|_{\Omega^*} &\leq 2V + 2(b-b') \max_{\substack{x, y \in X \\ x < y}} \frac{|g(x, y)|}{(x+a)(b-y)} \\ &\leq 2V + 2(b-b') \frac{G}{(-a'+a)(b-b')} = 2V + \frac{2G}{a-a'}. \end{aligned}$$

This is still not the desired bound for $\|f \cdot \mu\|_{\Omega^*}$, since a' depends on Ω . However the symmetric argument just mentioned can be carried out to give us $b-b'$ in place of $a-a'$ in the above, enabling us to apply (5.6). In fact, we have

$$\begin{aligned}\|f \cdot \mu\|_{\Omega^*} &\leq 2V + \min\left(\frac{2G}{a-a'}, \frac{2G}{b-b'}\right) \\ &= 2V + \frac{2G}{\max(a-a', b-b')} \leq 2V + \frac{4G}{\min(a, b)}.\end{aligned}$$

Since $\|f \cdot \mu\|_{\Omega} \leq \|f \cdot \mu\|_{\Omega^*}$, the proof of Proposition 5.5 is complete.

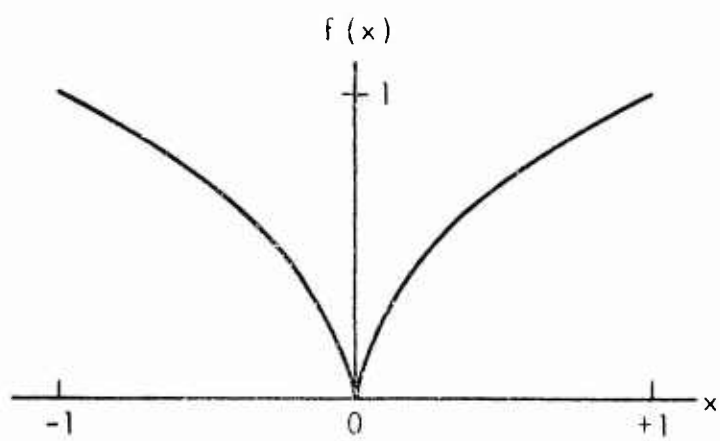
Example 5.7. Let the underlying space be the interval $[-1, 1]$, with its Borel subsets; let $\mu(S) = \int_S \operatorname{sgn} x \, dx$; and let $v(S) = \sqrt{|\mu(S)|}$. (See Fig. 2.) Then Proposition 5.5 tells us at once that $v \notin BV$, because of the behavior of f at 0.

To see this directly, merely define the chain $\Omega = \{\emptyset, S_1, \dots, S_{2k}\}$ by $S_{2j-1} = [-(j-1)/k, j/k]$ and $S_{2j} = [-j/k, j/k]$. Then $\|v\|_{\Omega} = 2\sqrt{k}$, which is unbounded.

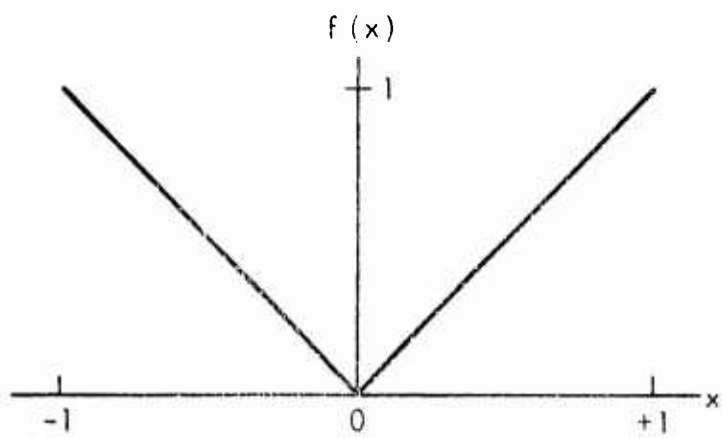
Example 5.8. Take μ as in Example 5.7, and let $v(S) = |\mu(S)|$. (See Fig. 2.) Clearly $v \ll \lambda$, where λ is Lebesgue measure; this shows that $v \in AC$, and hence that $v \in BV$. But we shall show that $v \notin pNA$.

The general idea of the proof which follows is to try to find a polynomial w in measures that approximates v in the variation norm. Such a w would behave "locally" like a measure plus a constant, just as a polynomial in several real variables behaves locally like a linear function in those variables (i. e., a homogeneous linear function plus a constant). On the other hand, in the neighborhood of

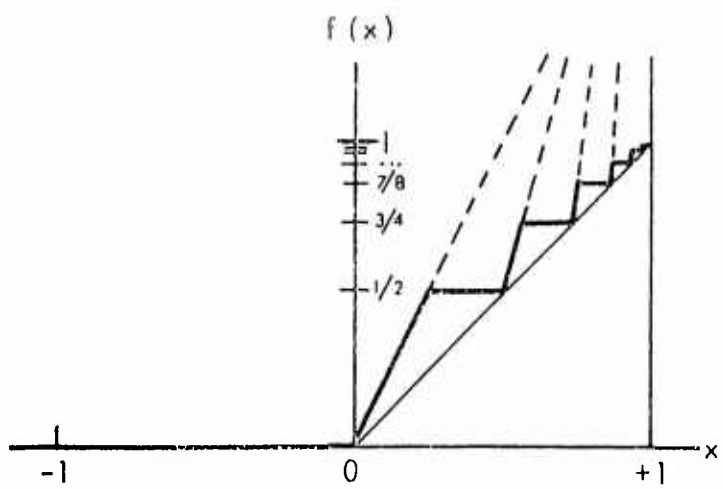
-50-



Example 5.7



Example 5.8



Example 5.9

Figure 2

sets $S_0 \subset I$ for which $\mu(S_0) = 0$, the game v does not behave at all like a measure plus a constant; this may be seen by considering, in such neighborhoods, both sets S with $\mu(S) > 0$ and sets S with $\mu(S) < 0$. Hence $\|v - w\|$ cannot be small, and the attempted approximation fails.

Proof for Example 5.8. Let w be any measure polynomial. Without loss of generality, $w(S) = p(v(S))$ where p is a polynomial in n variables, for some n , and $v = (v_1, \dots, v_n)$ is a vector of measures in NA^+ . For each i , we may write $v_i = \xi_i + \zeta_i$, where ξ_i and ζ_i are both in NA^+ and are respectively absolutely continuous and singular, with respect to Lebesgue measure λ . Let $I^* = I \setminus (I_1 \cup \dots \cup I_n)$, where each $I_i \subset I$ is such that $\zeta_i(I_i) = \zeta_i(I)$ and $\lambda(I_i) = 0$. Then for all Borel sets $S \subset I$ we have $\lambda(S \cap I^*) = \lambda(S)$ and, for each i , $\zeta_i(S \cap I^*) = 0$ and $v_i(S \cap I^*) = \xi_i(S)$.

Now choose a number k , and define the $2k$ sets

$$\begin{cases} T_j = (\frac{j-1}{k}, \frac{j}{k}) \cap I^*, & j = 1, \dots, k, \\ U_j = (-\frac{j}{k}, -\frac{j-1}{k}) \cap I^*, & j = 1, \dots, k. \end{cases}$$

These are pairwise disjoint, and partition I^* (and I) except for a set of measure zero. Next define a chain $\mathcal{C} = \{S_0, \dots, S_{2k}, I\}$ by

$$\begin{cases} S_0 = \emptyset \\ S_{2j} = \bigcup_{i=1}^j (T_i \cup U_i), & j = 1, \dots, k, \\ S_{2j+1} = S_{2j} \cup T_{j+1}, & j = 0, \dots, k-1. \end{cases}$$

Note that the values of v on this chain alternate between 0 and $1/k$.

Accordingly, we have

$$\begin{aligned} \|w - v\| &\geq \|w - v\|_C \\ &\geq \sum_{j=1}^k \left| w(S_{2j-1}) - w(S_{2j-2}) - \frac{1}{k} \right| + \sum_{j=1}^k \left| w(S_{2j}) - w(S_{2j-1}) + \frac{1}{k} \right|. \end{aligned}$$

By the mean value theorem we have

$$w(S_{2j-1}) - w(S_{2j-2}) = \xi(T_j) \cdot q(x_j),$$

where q denotes the gradient of p and x_j is some point on the line between $\pi(S_{2j-1})$ and $\pi(S_{2j-2})$. Similarly, we have

$$w(S_{2j}) - w(S_{2j-1}) = \xi(U_j) \cdot q(y_j),$$

where y_j is between $\pi(S_{2j})$ and $\pi(S_{2j-1})$. Hence

$$\begin{aligned} \|v - v\| &\geq \sum \left| \xi(T_j) \cdot q(x_j) - \frac{1}{k} \right| + \sum \left| \xi(U_j) \cdot q(y_j) + \frac{1}{k} \right| \\ &\geq \sum \left(-\xi(T_j) \cdot q(x_j) + \frac{1}{k} \right) + \sum \left(\xi(U_j) \cdot q(y_j) + \frac{1}{k} \right) \\ &= 2 - \sum \xi(T_j) \cdot q(x_j) + \sum \xi(U_j) \cdot q(y_j). \end{aligned}$$

If we begin instead with the chain $\mathcal{C}' = \{S'_0, \dots, S'_{2k}, I\}$, defined by

$$\begin{cases} S'_{2j} = S_{2j}, & j = 0, \dots, k, \\ S'_{2j+1} = S_{2j} \cup U_{j+1}, & j = 0, \dots, k-1, \end{cases}$$

the exactly analogous argument yields

$$\|w - v\| \geq 2 - \sum \xi(U_j) \cdot q(y'_j) + \sum \xi(T_j) \cdot q(x'_j).$$

Here x'_j is between $\xi(S'_{2j})$ and $\xi(S'_{2j-1})$ and y'_j is between $\xi(S'_{2j-1})$ and $\xi(S'_{2j-2})$. Combining the two estimates, we obtain

$$\begin{aligned} 2\|w - v\| &\geq 4 - \sum \xi(T_j) \cdot [q(x_j) - q(x'_j)] - \sum \xi(U_j) \cdot [q(y'_j) - q(y_j)] \\ &\geq 4 - n\|\xi(I^*)\|Q, \end{aligned}$$

where Q is the maximum of the norms of all the expressions in square brackets.

To complete the argument, suppose k is large. Then by the absolute continuity of the ξ_i , $\|\xi(T_j)\|$ and $\|\xi(U_j)\|$ will be small (uniformly in j). The distances $\|x_j - x'_j\|$ and $\|y'_j - y_j\|$ are also small, as each is bounded by $\|\xi(T_j)\| + \|\xi(U_j)\|$ (this may be seen by applying the triangle inequality to the parallelogram with vertices $\xi(S_{2j})$, $\xi(S_{2j-1})$, $\xi(S_{2j-2})$, and $\xi(S'_{2j-1})$). Thus, finally, applying the continuity of q (i.e., the continuous differentiability of p), we see that Q itself can be made arbitrarily small. Hence $\|w - v\| \geq 2$, and v is not in pNA .

Example 5.9. Take u as in Example 5.7, and take f to be a monotonic, piecewise linear function as illustrated in Fig. 2. If the rising segments are given slopes $2, 8, 24, \dots, k2^k, \dots$, then $|g|$ is unbounded. (Graphically, this means that the extensions of these segments (dashed lines) would cross the vertical $x = 1$ at arbitrarily high points.) Thus $f \cdot \mu$ is not in BV, by Proposition 5.5. Note that f could be made continuously differentiable, by "rounding off the corners" of the graph, without affecting this conclusion.

On the other hand, we could give the rising segments slopes of $2, 4, 8, \dots, 2^k, \dots$ and $|g|$ would be bounded (the dashed lines would all pass below the point $(1, 3)$), despite the unbounded derivative of f . In this case, therefore, $f \cdot \mu$ is in BV. It is not in pNA, however, because of the discontinuities in the derivative of f . (Compare Example 5.8.) But if the corners were rounded, as above, it is our conjecture that the result would be in pNA, even though the function f cannot be approximated in $C^1(R)$ by polynomials.*

More generally, we conjecture that a necessary and sufficient condition that $f \cdot \mu$ be in pNA (when μ is allowed to take both positive and negative values) is that $f \cdot \mu$ be in BV and f be continuously differentiable at each point in the interior of the range R of μ . **

* See Sec. 7.

** The multidimensional case is discussed at the end of Sec. 9.

6. THE VALUE OF SCALAR MEASURE GAMES

It is convenient to introduce the term normalized bv function for a function f in bv such that $f(1) = 1$.

PROPOSITION 6.1. Let Q be a
symmetric subspace of BV , and let φ be a value
on Q . If f is a normalized bv function and μ a
non-atomic probability measure such that $f \cdot \mu \in Q$,
then

$$\varphi(f \cdot \mu) = \mu.$$

For the proof we need the following

LEMMA 6.2. If μ is a non-atomic measure
on the measurable space $([0, 1], \mathcal{B})$, then there is
an automorphism Φ of $([0, 1], \mathcal{B})$ such that

$$\Phi_* \mu = \lambda,$$

where λ is Lebesgue measure.

Proof. The existence of such an automorphism is well-known, but it is easier to prove it from scratch than to deduce it from any of the far more general theorems that are explicitly available (see for example [H-N]). Define a function ψ from $[0, 1]$ onto itself by $\psi(x) = \mu([0, x])$. Because μ is nonnegative and non-atomic, ψ is nondecreasing and continuous; in particular $\psi^{-1}(y)$ is a point or a

nondegenerate closed interval for each y . Suppose now that

$y \in [0, 1]$ and that x is the largest point in $\psi^{-1}(y)$. Then

$\psi^{-1}[0, y] = [0, x]$, and so

$$\mu(\psi^{-1}[0, y]) = \mu([0, x]) = \psi(x) = y = \lambda([0, y]).$$

So

$$(6.3) \quad \mu(\psi^{-1}S) = \lambda(S)$$

when S is of the form $[0, y]$, and so for all S in \mathcal{B} . Now the $\psi^{-1}(y)$ are disjoint for distinct y , so $\psi^{-1}(y)$ can be an interval for at most denumerably many y . If there are no such y , we may set $\phi = \psi^{-1}$, and the existence of the desired automorphism is established. If there are such y , let T_0 be an uncountable Borel subset of $[0, 1]$ with $\lambda(T_0) = 0$, and such that T_0 contains all y for which $\psi^{-1}(y)$ is a nondegenerate interval. Then $\psi^{-1}(T_0)$ is also an uncountable Borel subset of $[0, 1]$, and from (6.3) it follows that its μ -measure is 0. Let ϕ_0 be any one-one transformation from T_0 onto $\psi^{-1}(T_0)$ that is Borel-measurable in both directions; it is well-known that such transformations exist (for example, see [M, p. 139, Corollary 1]). Define the transformation ϕ by .

$$\phi(y) = \begin{cases} \psi^{-1}(y) & \text{for } y \in [0, 1] \setminus T_0 \\ \phi_0(y) & \text{for } y \in T_0. \end{cases}$$

Then $\mu(\phi S) = \lambda(S)$, ϕ is one-one and onto, and ϕ takes Borel sets to Borel sets. Thus to complete the proof that ϕ is the desired automorphism, it is only necessary to show that ϕ^{-1} takes Borel sets to Borel sets, i.e., that if S is a Borel subset of $[0, 1] \setminus T_0$, then $\psi(S)$ is Borel. This follows from the fact that ψ is Borel-measurable (since it is continuous), and from a known theorem according to which a one-one Borel measurable function from a Borel set into $[0, 1]$ has a Borel range [M, p. 139, Theorem 3.2]. This completes the proof of the lemma.

Proof of Proposition 6.1. We assume that $(I, \mathcal{C}) = ([0, 1], \mathcal{B})$, as we may do without loss of generality by assumption (2.1). Consider first the case in which μ is Lebesgue measure λ . If S_1 and S_2 are any two sets of equal λ -measure, let Θ be a λ -measure preserving automorphism of (I, \mathcal{B}) such that the symmetric difference $(\Theta S_1 \setminus S_2) \cup (S_2 \setminus \Theta S_1)$ is of λ -measure 0 [H₂, p. 74]. Let $v = f \cdot \mu = f \cdot \lambda$. Then any coalition of λ -measure 0 is a null coalition, and hence by the footnote on page 9, has value 0. Hence $(\phi v)(\Theta S_1) = (\phi v)(S_2)$. But since Θ is measure preserving, we have for all S that $\Theta_* v = v$. Hence, by condition (2.2), we obtain

$$(\phi v)(S_2) = (\phi v)(\Theta S_1) = (\Theta_* \phi v)(S_1) = (\phi \Theta_* v)(S_1) = (\phi v)(S_1).$$

Hence ϕv coincides on any two sets of equal λ -measure. Since $(\phi v)(I) = f(\lambda(I)) = f(1) = 1$, it follows that $(\phi v)(S) = \lambda(S)$ when $\lambda(S)$ is the reciprocal of an integer, and hence also when it is rational. When f —and therefore also v and ϕv —are monotonic, it follows

(by approximating $\lambda(S)$ from above and below by rationals) that $(\varphi v)(S) = \lambda(S)$ for all S ; in general, we may express f as the difference of monotonic functions and get the same result.

In the general case, when $\mu \neq \lambda$, let Φ be the automorphism of Lemma 6.2. Then $\Phi_*(f \cdot \mu) = f \cdot (\Phi_* \mu) = f \cdot \lambda$, and so by what we have just proved, $\varphi \Phi_*(f \cdot \mu) = \varphi(f \cdot \lambda) = \lambda$. Hence by (2.2) and Lemma 6.2,

$$\varphi(f \cdot \mu) = \Phi_*^{-1} \Phi_* \varphi(f \cdot \mu) = \Phi_*^{-1} \varphi \Phi_*(f \cdot \mu) = \Phi_*^{-1} \lambda = \mu,$$

and the proof of the proposition is complete.

The following is a converse to Proposition 6.1:

PROPOSITION 6.4. Let Q be a closed symmetric subspace of BV . Let F be a set of normalized bv functions, such that Q is spanned by the set of all games $f \cdot \mu$, where $f \in F$ and μ is a non-atomic probability measure. Let φ be a linear mapping from Q into FA with $\|\varphi\| = 1$, such that $\varphi(f \cdot \mu) = \mu$ whenever f and μ are as in the previous sentence. Then φ is a value on Q .

Proof. Let $v = f \cdot \mu$, where f and μ are as stated. Then for each Θ in \mathcal{G} , $\Theta_* \mu$ is a non-atomic probability measure; so from the hypothesis it follows that

$$\varphi \Theta_* v = \varphi(f \cdot \Theta_* \mu) = \Theta_* \mu = \Theta_* \varphi v.$$

Since both φ and Θ_* are continuous, it follows that $\varphi \Theta_* - \Theta_* \varphi$ is a

continuous linear operator on Q that vanishes on a spanning subset.

Therefore it vanishes on all of Q , establishing condition (2. 2).

Similarly, the mapping that takes v to $(\omega v)(I) - v(I)$ is a continuous linear functional on Q that vanishes on a spanning subset, and so

on all of Q , thus establishing condition (2. 3). Finally, from

Proposition 4. 3 it follows that ω is positive. This completes the proof of the proposition.

7. THE VALUE ON pNA

In this section we will be chiefly concerned with paving the way for the proof of Theorem B. * We will not actually be able to prove Theorem B as stated, since it refers to "the value on $bv'NA$," whose existence we have not yet proved. What we will do is prove the existence of a unique value on pNA , and then prove Theorem B with " pNA " substituted for " $bv'NA$ ". Theorem B as stated will then follow easily once we have proved Theorem A (in the next section).

PROPOSITION 7.1. Let $\mu = (\mu_1, \dots, \mu_n)$ be a vector of measures in NA , and let f with $f(0) = 0$ be continuously differentiable on the range R of μ .

Then $f \cdot \mu \in pNA$.

The proof of the proposition is accomplished in several stages. First, let f be a polynomial in n variables.

LEMMA 7.2. Let $\mu = (\mu_1, \dots, \mu_n)$ be a vector of measures in NA , and let f be a polynomial in n variables with $f(0) = 0$. Then $f \cdot \mu$ is a linear combination of positive integer powers of non-atomic probability measures, and so in particular is in pNA .

Proof. We first prove the formula

* See page 18.

$$\begin{aligned}
 (7.3) \quad k! x_1 \dots x_k &= (x_1 + \dots + x_k)^k \\
 &\quad - \sum_{1 \leq i \leq k} (x_1 + \dots + x_k - x_i)^k \\
 &\quad + \sum_{1 \leq i < j \leq k} (x_1 + \dots + x_k - (x_i + x_j))^k \\
 &\quad - + \dots
 \end{aligned}$$

Indeed, the right side vanishes when $x_1 = 0$, and so it is divisible by x_1 ; similarly it is divisible by x_2, \dots, x_k . Therefore, since it is of degree k , it must be a multiple of $x_1 \dots x_k$. But the term $x_1 \dots x_k$ only arises from the first term on the right side, and there its coefficient is $k!$, so (7.3) is proved.

From (7.3) it follows that every polynomial in n variables is a linear combination of powers of partial sums of the variables. Since every signed measure is the difference of two nonnegative measures, every polynomial in measures is a polynomial in nonnegative measures. Hence every polynomial in measures is a linear combination of powers of nonnegative measures, and so, after appropriate normalization, it is a linear combination of powers of probability measures. This completes the proof of Lemma 7.2.

The proof of Proposition 7.1 proceeds by an argument involving approximation to a general f by polynomials. First, note that we may assume that R is of full dimension, i. e., that

it has an interior in E^n . If not, let m be the true dimension of R ; then because $(0, \dots, 0) \in R$, we may find a linear transformation ψ of E^n onto E^m which is 1-1 on R ; let $\Theta = \psi|_R$. Now define a vector measure ξ of dimension m by $\xi = \Theta_\# \mu$, a function g on ΘR by $g(x) = f(\Theta^{-1}x)$, and a set function w by $w(S) = g(\xi(S))$; then $w = v$. Since the range of ξ is ΘR , which is of full dimension in E^m , the reduction is complete.

Now for continuous functions f on R , define

$$\|f\|_0 = \max_{x \in R} |f(x)|.$$

Let $C^1 = C^1(R)$ be the Banach space of continuously differentiable functions on R , with the norm

$$\|f\|_1 = \|f\|_0 + \sum_i \|f_i\|_0,$$

where $f_i = \partial f / \partial x_i$ in the interior of R , and is the appropriate continuous extension on the boundary.

LEMMA 7.4. The polynomials are dense in C^1 .

Proof. This is proved in [C-H], Chapter II, Sec. 4, Subsection 3, p. 68, for the case in which R is an n -dimensional cube. The general case may be reduced to this one by imbedding R in a cube R' and extending f and its derivatives in an arbitrary way from R to R' , so that the extended function is in $C^1(R')$. The existence of such an extension is well-known (see, for example, [W]).

To complete the proof of Proposition 7.1, let $\mu_i = \xi_i - \zeta_i$, where ξ_i and ζ_i are nonnegative measures. Let g be any member of C^1 , and let g' be the vector of its partial derivatives. Let $S_1 \subset S_2 \subset \dots$ be a chain such that $\mu(S_i)$ is in the interior of R for each i . Then by the mean value theorem,

$$\begin{aligned} & \sum_j |g(\mu(S_{j+1})) - g(\mu(S_j))| \\ &= \sum_j |\mu(S_{j+1} \setminus S_j) \cdot g'(\mu(S_j) + \theta_j \mu(S_{j+1} \setminus S_j))| \\ &\leq \sum_j \sum_{i=1}^n |\mu_i(S_{j+1} \setminus S_j)| \|g\|_1 \\ &\leq \|g\|_1 \sum_{i=1}^n [\xi_i(I) + \zeta_i(I)]. \end{aligned}$$

Hence if u is defined by $u(S) = g(\mu(S))$, then

$$(7.5) \quad \|u\| \leq \|g\|_1 \sum_{i=1}^n [\xi_i(I) + \zeta_i(I)],$$

since the demand that $\mu(S_i)$ be in the interior of R cannot decrease the supremum of the expression defining the norm.

Now approximate to f in C^1 by a polynomial p , and let $w(S) = p(\mu(S))$. Then from (7.5) it follows that

$$\|w - v\| \leq \|p - f\|_1 \sum_{i=1}^n [\xi_i(I) + \zeta_i(I)].$$

Since the right side can be made arbitrarily small, so can the left side, and the proof of Proposition 7.1 is complete.

PROPOSITION 7.6. There is a value φ on
pNA, with $\|\varphi\| = 1$. Furthermore, let v in pNA
be such that there exist μ , f , and U as follows:

(7.7) μ is a vector of non-atomic measures with
range R , f is a real-valued function defined
on R and continuously differentiable there,
 U is a convex neighborhood in R
of the diagonal $[0, \mu(I)]$,

and

$v(S) = f(\mu(S))$ whenever $\mu(S) \in U$.

Then

$$\varphi(v)(S) = \int_0^1 f_{\mu(S)}(t\mu(I))dt,$$

where $f_{\mu(S)}$ is the derivative of f in the direction
 $\mu(S)$.

Proof. Let v , μ , f , and U be as above. Define a signed
measure v by

$$v(S) = \int_0^1 f_{\mu(S)}(t\mu(I))dt.$$

Probably the easiest way to verify the complete additivity of ν is first to note that when R has full dimension, it follows at once from the explicit formula in terms of partial derivatives (3.2), and then to reduce the general case to that of full dimension by arguments similar to those following (3.2); indeed, for g and ξ as defined in the proof of Proposition 7.1 it is easily verified that

$$\int_0^1 g_{\xi}(S)(t \xi(I)) dt = \int_0^1 f_{\mu}(S)(t \mu(I)) dt.$$

Let $I = S^+ \cup S^-$ be a Hahn decomposition [H_1 , p. 121] of I with respect to ν ; that is, ν is nonnegative on S^+ and its subsets, nonpositive on S^- and its subsets, and $S^+ \cap S^- = \emptyset$. Then

$$\|\nu\| = |\nu(S^+)| + |\nu(S^-)|.$$

Let m be an arbitrary positive integer. Because of Lyapunov's theorem, it is possible to partition S^+ into disjoint sets S_1^+, \dots, S_m^+ such that $\mu(S_j^+) = \mu(S^+)/m$ for all j , and similarly to partition S^- into disjoint sets S_1^-, \dots, S_m^- such that $\mu(S_j^-) = \mu(S^-)/m$ for all j .

Now define a chain

$$S_0 \subset S_1 \subset \dots \subset S_{2m}$$

by

$$S_{2j} = (S_1^+ \cup S_1^-) \cup \dots \cup (S_j^+ \cup S_j^-)$$

$$S_{2j+1} = S_{2j} \cup S_{j+1}^+.$$

Let $y = \mu(S^+)$ and $b = \mu(I)$, so $\mu(S^-) = b - y$. Then

$$\mu(S_{2j}) = \frac{jb}{m},$$

$$\mu(S_{2j+1}) = \frac{jb+y}{m},$$

and hence we may choose m sufficiently large so that $\mu(S_\ell) \in U$ for all ℓ . Hence

$$\begin{aligned} (7.8) \quad \|v\| &\geq \sum_{\ell=1}^{2m} |f(\mu(S_\ell)) - f(\mu(S_{\ell-1}))| \\ &= \sum_{j=0}^{m-1} |f(\frac{jb+y}{m}) - f(\frac{jb}{m})| + \sum_{j=0}^{m-1} |f(\frac{(j+1)b}{m}) - f(\frac{jb+y}{m})| \\ &\geq \left| \sum_{j=0}^{m-1} \{f(\frac{jb+y}{m}) - f(\frac{jb}{m})\} \right| + \left| \sum_{j=0}^{m-1} \{f(\frac{(j+1)b}{m}) - f(\frac{jb+y}{m})\} \right|. \end{aligned}$$

If we look at $f(\frac{jb}{m} + \theta y)$ as a function of θ , then an application of the mean value theorem yields

$$(7.9) \quad f(\frac{jb+y}{m}) - f(\frac{jb}{m}) = \frac{1}{m} f_y(\frac{jb}{m} + \tau y),$$

where $0 \leq \tau \leq \frac{1}{m}$. Further, condition (7.7) implies that f_y is uniformly continuous in R ; so the right side of (7.9) is

$$= \frac{1}{m} f_y(\frac{jb}{m}) + o(\frac{1}{m}),$$

as $m \rightarrow \infty$, where the $o(\frac{1}{m})$ is uniform in j . Similarly, we have

$$f(\frac{(j+1)b}{m}) - f(\frac{jb+y}{m}) = \frac{1}{m} f_{b-y}(\frac{jb}{m}) + o(\frac{1}{m}).$$

Applying these remarks to (7.8), we get

$$\|v\| \geq \left| \sum_{j=0}^{m-1} \frac{1}{m} f_y \left(\frac{jb}{m} \right) \right| + \left| \sum_{j=0}^{m-1} \frac{1}{m} f_{b-y} \left(\frac{jb}{m} \right) \right| + o(1).$$

Note that the sums are approximating sums to a Riemann integral; and since f_y is continuous near—and on—the diagonal, this integral exists. Hence by going to the limit we obtain

$$\begin{aligned} (7.10) \quad \|v\| &\geq \left| \int_0^1 f_y(tb) dt \right| + \left| \int_0^1 f_{b-y}(tb) dt \right| \\ &= |v(S^+)| + |v(S^-)| = \|v\|. \end{aligned}$$

Now define

$$\varphi v = v.$$

We must prove that this is an admissible definition, i. e., that it does not depend on the choice of μ , f , and U in (7.7). Indeed, suppose ξ , g , and V satisfy (7.7), and that $v(S) = g(\xi(S))$ whenever $\xi(S) \in V$. Define a set function w by $w(S) = 0$ for all S , a vector measure ζ by $\zeta = (\mu, \xi)$, a function h by $h(x, y) = f(x) - g(y)$, and a compact convex neighborhood W of the diagonal $[0, \xi(I)]$ by $W = U \times V$. Then ζ , h , and W satisfy condition (7.7), and whenever $\zeta(S) \in W$, we have

$$h(\zeta(S)) = f(\mu(S)) - g(\xi(S)) = v(S) - v(S) = 0 = w(S).$$

Let

$$\eta(S) = \int_0^1 g_{\xi(S)}(t\xi(I))dt$$

$$\sigma(S) = \int_0^1 h_{\zeta(S)}(t\zeta(I))dt.$$

It is then easily verified that $\sigma = \nu - \eta$. Applying (7.10) to w , we deduce

$$\|\nu - \eta\| = \|\sigma\| \leq \|w\| = \|0\| = 0.$$

So $\nu - \eta = 0$, i.e., $\nu = \eta$. This proves that φ is well-defined.

Let Q be the set of set functions ν in pNA satisfying the second sentence of our proposition. Q contains all the linear combinations of powers of measures, so it is dense in pNA . We have already defined φ on Q ; it is easily verified that Q is a linear subspace of pNA , and that φ is linear on Q . From (7.10) it follows that $\|\varphi \nu\| \leq \|\nu\|$ for $\nu \in Q$, so φ is continuous on Q and $\|\varphi\| \leq 1$; from $\varphi \lambda = \lambda$ we deduce $\|\varphi\| \geq 1$, so $\|\varphi\| = 1$. From the completeness of FA it then follows that φ can be uniquely extended to be a continuous linear operator from pNA to FA , and this extension will also have norm 1.

It remains to verify that φ is indeed a value, and for this we will use Proposition 6.4. Thus let $\nu = \mu^k$, where $\mu \in NA$ is a probability measure. Then using formula (3.2), we get

$$\begin{aligned}
 (\varphi v)(S) &= \mu(S) \int_0^1 \left(\frac{d}{dt} t^k \right) dt \\
 &= \mu(S) [1^k - 0^k] \\
 &= \mu(S),
 \end{aligned}$$

and the proof of Proposition 7.6 is complete.

PROPOSITION 7.11. The value on pNA is
unique.

Before proving this proposition, we require a number of lemmas.

LEMMA 7.12. There is a set \mathcal{U} of
probability measures in NA with the following
properties:

- (7.13) Any two distinct members of \mathcal{U} are mutually
singular.
- (7.14) For any measure ν in NA, there is a sequence
 $\{\mu_1, \mu_2, \dots\}$ of measures in \mathcal{U} such that
$$\nu \ll \sum_{i=1}^{\infty} \mu_i / 2^i.$$

Proof. For $\alpha, \beta \in NA^+$, let

$$\alpha = \alpha_{\beta}^{ac} + \alpha_{\beta}^{\perp}$$

denote the canonical decomposition of α into measures that are, respectively, absolutely continuous and singular w. r. t. β . It may be verified* that

$$(7.15) \quad (\alpha_1 + \alpha_2)_{\beta}^{ac} = (\alpha_1)_{\beta}^{ac} + (\alpha_2)_{\beta}^{ac}$$

and that

$$(7.16) \quad \beta_1 \perp \beta_2 \Rightarrow \alpha_{\beta_1}^{ac} \perp \alpha_{\beta_2}^{ac}.$$

Recall also [H_1 , p. 127, exercise 10] that

$$(7.17) \quad \beta_1 \perp \beta_2, \beta_1 \perp \beta_3 \Rightarrow \beta_1 \perp \beta_2 + \beta_3.$$

Let \mathcal{U} be a set of probability measures in NA that is maximal w. r. t. property (7.13); such a set exists by Zorn's lemma. To prove (7.14), let ν in NA^+ be given. Let \mathcal{V} be the set of all measures μ in \mathcal{U} that are not mutually singular to ν , i. e., such that

$$\nu_{\mu}^{ac}(I) > 0.$$

Let $\{\mu_1, \dots, \mu_k\}$ be any finite subset of \mathcal{V} . Setting $\zeta = \mu_1 + \dots + \mu_k$ we obtain from (7.16) and (7.17) that

$$\nu(S) \geq \nu_{\zeta}^{ac}(S) = \nu_{\mu_1}^{ac}(S) + \dots + \nu_{\mu_k}^{ac}(S)$$

for all $S \subset I$. Since this holds for any finite subset $\{\mu_1, \dots, \mu_k\}$

* E. g., by looking at "supports" of the measures involved.

of \mathcal{V} , it follows that \mathcal{V} is denumerable; furthermore, it follows that

$$\sum_{\mu \in \mathcal{V}} v_{\mu}^{\text{ac}}(S) \leq v(S),$$

and so in particular the series on the left converges. From this and Proposition 4.4 it follows that if we define

$$(7.18) \quad \xi = \sum_{\mu \in \mathcal{V}} v_{\mu}^{\text{ac}},$$

then $\xi \in \text{NA}$, and $\xi(S) \leq v(S)$ for S . Hence if we define

$$\eta = v - \xi,$$

then $\eta \in \text{NA}^+$.

Suppose that $\eta \neq 0$; then there is a member θ of \mathcal{U} such that

$$\eta_{\theta}^{\text{ac}}(I) > 0.$$

Now $v = \xi + \eta$; hence by (7.15),

$$v_{\theta}^{\text{ac}}(I) = \xi_{\theta}^{\text{ac}}(I) + \eta_{\theta}^{\text{ac}}(I) \geq \eta_{\theta}^{\text{ac}}(I) > 0.$$

Hence $\theta \in \mathcal{V}$. But then

$$\xi = v_{\theta}^{\text{ac}} + \sum_{\mu \in \mathcal{V} - \{\theta\}} v_{\mu}^{\text{ac}} = v_{\theta}^{\text{ac}} + \gamma,$$

say. Hence by (7.15),

$$\xi_{\theta}^{\text{ac}} = (v_{\theta}^{\text{ac}})_{\theta}^{\text{ac}} + \gamma_{\theta}^{\text{ac}} = v_{\theta}^{\text{ac}} + \gamma_{\theta}^{\text{ac}}.$$

Hence

$$v_{\theta}^{\text{ac}}(I) = \xi_{\theta}^{\text{ac}}(I) + \eta_{\theta}^{\text{ac}}(I) > \xi_{\theta}^{\text{ac}}(I) = v_{\theta}^{\text{ac}}(I) + \gamma_{\theta}^{\text{ac}}(I) \geq v_{\theta}^{\text{ac}}(I).$$

This absurdity proves that $\eta \equiv 0$, hence $v = \xi$. Assertion (7.14)

now follows from (7.18) when $v \in NA^+$; and the general case (i. e., $v \in NA$) follows from this without difficulty.

LEMMA 7.19. There is a family \mathcal{J} of non-atomic probability measures with the following properties:

(7.20) \mathcal{J} spans NA .

(7.21) For any finite subset \mathcal{K} of \mathcal{J} , there is a finite subset \mathcal{Y} of \mathcal{J} , such that the members of \mathcal{Y} are mutually singular and each member of \mathcal{K} is a linear combination of members of \mathcal{Y} .

Proof. Let us define a dyadic interval to be an interval of the form

$$J(k, l) = [k/2^l, (k+1)/2^l],$$

where $0 \leq k < 2^l$. Denote the set of all dyadic intervals by \mathcal{J} .

Let \mathcal{U} be as in Lemma 7.12. For each $\mu \in \mathcal{U}$, define a family of sets $\{S_\mu^\alpha : 0 \leq \alpha \leq 1\}$ in accordance with Lemma 5.4; i. e., so that

$$\mu(S_\mu^\alpha) = \alpha$$

and $\alpha > \beta$ implies $S_\mu^\alpha \supset S_\mu^\beta$. If $J = [\beta, \alpha]$ is a dyadic interval, define

$$T_\mu^J = S_\mu^\alpha \setminus S_\mu^\beta;$$

clearly $\mu(T_\mu^J) = \lambda(J) = \beta - \alpha$. For each $J \in \mathcal{J}$, define a measure μ^J on I by

$$\mu^J(S) = \mu(S \cap T_\mu^J) / \lambda(J);$$

then μ^J is a non-atomic probability measure. Define

$$\mathcal{Z} = \{\mu^J\}_{\mu \in \mathcal{U}, J \in \mathcal{J}}.$$

We must prove that \mathcal{Z} obeys (7.20) and (7.21).

For each $\mu \in \mathcal{U}$, let

$$\mathcal{Z}_\mu = \{\mu^J\}_{J \in \mathcal{J}}:$$

First we prove (7.21). Assume, for a start, that $\mathcal{X} \subset \mathcal{Z}_\mu$ for some fixed μ ; let

$$\mathcal{X} = \{\mu^{J_1}, \dots, \mu^{J_m}\},$$

where

$$J_1 = J(k_1, l_1), \dots, J_m = J(k_m, l_m).$$

Let

$$l = \max(l_1, \dots, l_m),$$

and define $\mathcal{Y} = \mathcal{Y}(\mathcal{X})$ by

$$\mathcal{Y} = \{\mu^{J(0, l)}, \mu^{J(1, l)}, \dots, \mu^{J(2^l - 1, l)}\}.$$

Then it may be verified that (7.21) holds.

In the general case, when \mathcal{X} is included in no one \mathcal{Z}_μ , we may define

$$\mathcal{X}_\mu = \mathcal{X} \cap \mathcal{J}_\mu$$

for each μ ; all but a finite number of the \mathcal{X}_μ are empty. Define $\gamma(\mathcal{X}_\mu)$ as above, and define $\gamma = \gamma(\mathcal{X})$ by

$$\gamma = \bigcup_\mu \gamma(\mathcal{X}_\mu),$$

where the union is taken over those μ for which \mathcal{X}_μ is nonempty. Again, (7.21) may be verified. This completes the proof of (7.21).

To prove (7.20), we must show that for every $\nu \in \mathcal{N}\mathcal{A}$ and $\epsilon > 0$, there is a linear combination ζ of members of \mathcal{J} , such that $\|\nu - \zeta\| < \epsilon$. For a start, let us assume that there is a member μ of \mathcal{A} such that $\nu \ll \mu$. By the Radon-Nikodym theorem, we may find a Borel-measurable μ -integrable function f such that

$$\nu(S) = \int_S f(t) d\mu(t)$$

for all $S \subset I$. Now f can be approximated by step functions in the norm of the space $L^1(I, \mathcal{C}, \mu)$, and hence by step functions in which each step is a dyadic interval; that is, f can be approximated in the above norm by linear combinations of characteristic functions of dyadic intervals. In other words, we can find such a function g with

$$\int_I |f(t) - g(t)| d\mu(t) < \epsilon.$$

Then if we define a measure ζ by

$$\zeta(S) = \int_S g(t) d\mu(t),$$

then ζ is a linear combination of members of \mathcal{J}_μ , and $\|\nu - \zeta\| < \epsilon$, as was to be proved.

In the general case, when there is no one μ such that $\nu \ll \mu$, use (7.14) to find a sequence $\{\mu_1, \mu_2, \dots\}$ such that

$$\nu \ll \sum_{i=1}^{\infty} \mu_i / 2^i.$$

Let $\nu = \nu_i + \tau_i$ be the unique decomposition of ν into measure ν_i and τ_i . Then $\sum_i \|\nu_i\|$ converges and

$$\nu = \sum_{i=1}^{\infty} \nu_i.$$

Now for each i we may find a linear combination ζ_i of members of \mathcal{J}_{μ_i} such that

$$\|\nu_i - \zeta_i\| < \epsilon / 2^{i+1};$$

then $\sum_i \|\zeta_i\|$ converges, and setting

$$\xi = \sum_{i=1}^{\infty} \zeta_i,$$

we conclude that

$$\|\nu - \xi\| < \epsilon / 2.$$

Now choose k sufficiently large so that

$$\sum_{i=k+1}^{\infty} \|\zeta_i\| < \epsilon / 2.$$

Then setting

$$\zeta = \sum_{i=1}^k \zeta_i,$$

we obtain

$$\|v - \zeta\| \leq \|v - \xi\| + \sum_{i=k+1}^{\infty} \|\zeta_i\| < \epsilon.$$

Since ζ is a finite linear combination of finite linear combinations of members of the various \mathcal{J}_{μ_i} , it is itself a finite linear combination of members of \mathcal{J} . This completes the proof of (7.20), and with it the proof of Lemma 7.19.

Let \mathcal{J} be as in Lemma 7.19, and let A be the set of all set functions of the form $f \cdot \mu$, where μ is a vector of mutually singular measures in \mathcal{J} and f is continuously differentiable on the range of μ (which is always the unit cube of dimension equal to that of μ).

LEMMA 7.22. A is a linear reproducing subspace of BV whose closure is pNA.

Proof. Clearly if $v \in A$ then $\alpha v \in A$ for all real α . Next, let $f \cdot \mu$ and $g \cdot v$ be in A , where $\mu = (\mu_1, \dots, \mu_n)$, $v = (v_1, \dots, v_m)$, the μ_i are mutually singular members of \mathcal{U} , the v_j are mutually singular members of \mathcal{U} , and f and g are continuously differentiable. Now apply (7.21) to the set

$$\mathcal{X} = \{\mu_1, \dots, \mu_n, v_1, \dots, v_m\}$$

Then we can find a finite subset

$$\mathcal{Y} = \{\tau_1, \dots, \tau_p\}$$

of \mathcal{U} such that each μ_i and each ν_j is a linear combination of members of \mathcal{Y} , i. e.,

$$\mu_i = \sum_{k=1}^p \alpha_{ik} \tau_k,$$

$$\nu_i = \sum_{k=1}^p \beta_{ik} \tau_k.$$

Now define h on $[0, 1]^p$ by

$$(7.23) \quad h(x_1, \dots, x_p) = f\left(\sum_{k=1}^p \alpha_{1k} x_k, \dots, \sum_{k=1}^p \alpha_{nk} x_k\right) + g\left(\sum_{k=1}^p \beta_{1k} x_k, \dots, \sum_{k=1}^p \beta_{mk} x_k\right).$$

Then if τ is the vector measure (τ_1, \dots, τ_p) , then $h \cdot \tau = f \cdot \mu + g \cdot \nu$.

Since h is continuously differentiable, it follows that A is a subspace.

Next, note that if in (7.23) we replace the $+$ sign by a multiplication sign, then h will still be continuously differentiable, and we will have $h \cdot \tau = (f \cdot \mu)(g \cdot \nu)$. This proves that A is an algebra (closed under multiplication). But then by Proposition 4.5 and the remark following it, \bar{A} is also an algebra. Now by (7.20), \bar{A} contains NA , hence since it is an algebra it must contain all powers of measures in NA ; therefore, since it is closed, it contains pNA . But since $A \subset pNA$, it follows that $\bar{A} \subset pNA$, hence $\bar{A} = pNA$.

Finally, to demonstrate that A is reproducing, let $f \cdot \mu \in A$, where f and μ satisfy the appropriate conditions. Let D be the maximum of the absolute values of the partial derivatives of f , and let n be the dimension of μ . Then both

$$f \cdot \mu + D \sum_{i=1}^n \mu_i$$

and

$$D \sum_{i=1}^n \mu_i$$

are monotonic, and

$$f \cdot \mu = (f \cdot \mu + D \sum_{i=1}^n \mu_i) - (D \sum_{i=1}^n \mu_i).$$

This completes the proof of Lemma 7.22.

LEMMA 7.24. For all $v \in A$,

$$\|v\| = \|v\|_A.$$

Proof. Represent $v \in A$ by $f \cdot \mu$, where μ is a vector of mutually singular measures in \mathcal{J} , with range $R = [0, 1]^n$, and $f \in C^1(R)$. Define f^+ on R by

$$f^+(x) = \sup \sum_{j=1}^m \max(0, f(y_j) - f(y_{j-1})),$$

taking the supremum over all finite sequences

$0 = y_0 \leq y_1 \leq \dots \leq y_m = x$. Define f^- similarly but with

$\max(0, f(y_{j-1}) - f(y_j))$ for the summand. Then f^+ and f^- are

continuous and nondecreasing, and $f^+ - f^- = f$. Moreover, we have

$$\|v\| = f^+(e) + f^-(e),$$

since the sequences $\{y_i\}$ above are exactly the sequences $\{\mu(S_j)\}$ that arise from the chains $\emptyset = S_0 \subset \dots \subset S_m = I$ that determine the variation norm. (Here the mutual singularity of the components of μ is crucial.) We are prevented from asserting similarly that $\|v\|_A = f^+(e) + f^-(e)$ only because f^+ and f^- may fail to be differentiable.* Our object in the following will be to find suitably differentiable substitutes for f^+ and f^- ; i.e., nondecreasing functions $h, k \in C^1(R)$ with $h - k = f$ and $h(0) = k(0) = 0$, and such that $h(e)$ and $k(e)$ are approximately equal to $f^+(e)$ and $f^-(e)$ respectively.**

Write f_i for $\partial f / \partial x_i$, and let $D = \max_i \max_x |f_i(x)|$. Fix $\epsilon > 0$, and let $\delta > 0$ be such that $\|x - y\| < \delta$ implies $\max_i |f_i(x) - f_i(y)| < \epsilon$, for all $x, y \in R$. We shall also require that $\delta < \epsilon / D$.

We now define a linear operator " $\#$ " on the continuous functions on R :

$$g^\#(x) = \int_{y \in R} g((1-\delta)x + \delta y) dy,$$

* A simple example for $n = 2$ is provided by $f(x) = x_1 + x_2 - 2x_1x_2$, in which case $f^+(x) = \max(x_1, x_2, f(x))$.

** Whether they can be made exactly equal, for all $f \in C^1([0, 1]^n)$, is an interesting open question. In the above example, we may take $h(x) = x_1 + x_2 - x_1x_2$ and $k(x) = x_1x_2$.

or, equivalently,

$$g^{\#}(x) = \frac{1}{\delta^n} \int \dots \int_{z_i = (1-\delta)x_i}^{(1-\delta)x_i + \delta} \dots \int g(z) dz_1 \dots dz_n.$$

(Note that the region over which this "moving average" is taken lies wholly within R .) From the second expression for $g^{\#}(x)$ it is apparent that $g^{\#} \in C^1(R)$, even if g is only continuous, and that if $g \in C^1(R)$ then $(g^{\#})_i = (1-\delta)g_i^{\#}$. From the first expression it is apparent that if g is nondecreasing then so is $g^{\#}$, and moreover

$$g^{\#}(0) \geq g(0) \text{ and } g^{\#}(e) \leq g(e).$$

Finally, since we are averaging a continuous function, for every $x \in R$ there is a $y \in R$ such that $\|x-y\| < \delta$ and $g(y) = g^{\#}(x)$.

Applying this to the derivatives of our original function f we obtain

$$|f_i^{\#}(x) - f_i(x)| < \epsilon$$

for all x and i . Hence we have

$$\begin{aligned} \left| \frac{\partial(f^{\#}-f)(x)}{\partial x_i} \right| &\leq |(f^{\#})_i(x) - f_i^{\#}(x)| + |f_i^{\#}(x) - f_i(x)| \\ &< \delta |f_i^{\#}(x)| + \epsilon \\ &\leq \delta D + \epsilon \\ &\leq 2\epsilon, \end{aligned}$$

which tells us that the function $f^{\#} - f + 2\epsilon u$ is nondecreasing,

where u is defined by $u(x) \equiv \sum_1^n x_i$.

Now define

$$h = f^{+\#} - f^{+\#}(0) + 2\epsilon u.$$

Then $h \in C^1(r)$ and $h(0) = 0$; also h is nondecreasing, being the sum of nondecreasing functions. Next define

$$k = h - f.$$

Clearly, $k \in C^1(R)$ and $k(0) = 0$. Moreover, we can express k as a sum of nondecreasing functions:

$$k = (f^+ - f)^{\#} + (f^{\#} - f + 2\epsilon u) - f^{+\#}(0),$$

using the linearity of " $\#$ ", so k too is nondecreasing. Thus, $h \cdot \mu$ and $k \cdot \mu$ are members of A^+ , and so we have $\|v\|_A \leq h(e) + k(e) = 2h(e) - f(e)$. Hence

$$\begin{aligned} \|v\| &\leq \|v\|_A \leq 2f^{+\#}(e) - 2f^{+\#}(0) + 4\epsilon u(e) - f(e) \\ &\leq 2f^+(e) + 4\epsilon n - f(e) \\ &= f^+(e) + f^-(e) + 4\epsilon n \\ &= \|v\| + 4\epsilon n. \end{aligned}$$

Since this holds for all $\epsilon > 0$, we have $\|v\| = \|v\|_A$, and the proof of Lemma 7.24 is complete..

We are now ready for the

Proof of Proposition 7.11. If φ is a value on pNA, then by Proposition 4.12 and Lemmas 7.22 and 7.24, it follows that φ

must be continuous. On the other hand, by Proposition 6.1 we have $\varphi \mu^k = \mu$ for all probability measures μ in NA, and all positive integers k . This determines φ on a spanning subset of pNA, and so by linearity and continuity on all of pNA. This completes the proof of Proposition 7.11.

We close this section with the following proposition, which is an easy consequence of Proposition 4.12 and Lemmas 7.22 and 7.24.

PROPOSITION 7.25. pNA is reproducing, and

$$\|v\| = \|v\|_{\text{pNA}}$$

for all $v \in \text{pNA}$.

A proposition related to, and in a sense generalizing Proposition 7.25 will be proved in Appendix B.

8. THEOREMS A AND B

In this section we will extend to all of $bv'NA$ the value that we defined on pNA in the previous section, thereby proving Theorem A and completing the proof of Theorem B. (Both theorems are stated in Sec. 3.) For this purpose we must first recall some facts and definitions concerning functions of bounded variation.

It will be convenient to deal not only with functions defined on $[0, 1]$, but with functions of bounded variation defined on any closed bounded interval. To avoid problems that would arise from adding functions having different domains, we will consider functions f of bounded variation on the entire real line, with the proviso that there exist real numbers c and d such that f is constant in $(-\infty, c]$ and in $[d, \infty)$; the interval $[c, d]$ is then called a support of f . The space of all such functions will be denoted bv^* , and the total variation of a bv^* function f will be denoted $\|f\|$. The real line $(-\infty, \infty)$ will be denoted E^1 .

A bv^* function f is said to be a left-continuous single-jump function if there is a real number s such that

$$f(t) = \begin{cases} 1 & \text{for } t > s \\ 0 & \text{for } t \leq s. \end{cases}$$

It is said to be a right-continuous single-jump function if there is a real number s such that

$$f(t) = \begin{cases} 1 & \text{for } t \geq s \\ 0 & \text{for } t < s. \end{cases}$$

In either case it is said to have a jump at s. A bv^* function f is a left-continuous (right-continuous) jump function if it is of the form $\sum_i \alpha_i f_i$, where all the f_i are left-continuous (right-continuous) single-jump functions, and $\sum_i \alpha_i$ is either a finite sum or an absolutely convergent infinite sum of real numbers. If the jump of f_i is at s_i , then we may assume without loss of generality that the s_i are all different, and that none of the α_i vanish. Then the set $\{s_i\}$ —the set of discontinuities of f —is called the spectrum of f and is denoted $\mathcal{J}(f)$; furthermore we have

$$\|f\| = \sum_i |\alpha_i|.$$

A function f in bv^* is said to be a jump-function if

$$f = f^+ + f^-,$$

where f^+ is a right-continuous jump function and f^- is a left-continuous jump function; the decomposition is essentially^{*} unique.

In this case the spectrum $\mathcal{J}(f)$ is defined to be $\mathcal{J}(f^+) \cup \mathcal{J}(f^-)$,

and we have

$$\|f\| = \|f^+\| + \|f^-\|.$$

^{*} Up to an additive constant.

Next, let f in bv^* be continuous. Then we may define a measure ν_f on E^1 by

$$\nu_f([s, t]) = f(t) - f(s),$$

whenever $s \leq t$. If $\|\nu_f\|$ denotes the total variation of ν_f , then it may be verified that

$$\|f\| = \|\nu_f\|.$$

It is easily verified that every function f in bv^* may be essentially* uniquely written as

$$f = f^c + f^- + f^+$$

where f^c is continuous, f^- is a left-continuous jump-function, and f^+ is a right-continuous jump-function; furthermore, we have

$$\|f\| = \|f^c\| + \|f^-\| + \|f^+\|.$$

For the measure corresponding to the continuous component f^c of f , we will write ν_f rather than the more cumbersome ν_{f^c} .

Two bv^* functions f and g are said to be mutually singular (written $f \perp g$) if ν_f and ν_g are mutually singular measures, $\mathcal{S}(f^+) \cap \mathcal{S}(g^+) = \emptyset$, and $\mathcal{S}(f^-) \cap \mathcal{S}(g^-) = \emptyset$. If $f \perp g$ then

$$(8.1) \quad \|f + g\| = \|f\| + \|g\|.$$

Clearly if $f \perp g$ then $f \perp \alpha g$ for all real α . Moreover

* Up to additive constants.

(8.2) if $f_1 \perp g$ and $f_2 \perp g$, then $f_1 + f_2 \perp g$;

this follows from the corresponding fact for mutually singular measures (cf. $[H_1]$, p. 127, exercise 10). Next, we have

Remark 8.3. If g_0, \dots, g_ℓ are bv^* functions such that $g_i \perp g_j$ for all i and j , then

$$\|g_0 + \dots + g_\ell\| = \|g_0\| + \dots + \|g_\ell\|;$$

this remark follows at once from multiple applications of (8.1) and (8.2).

A bv^* function f is said to be singular if $\nu_f \perp \lambda$, where λ is Lebesgue measure; this is equivalent to saying that $f \perp g$ where g is the identity (i.e., $g(t) = t$) on a support of f .

A bv^* function f is said to be absolutely continuous w.r.t. a bv^* function g (written $f \ll g$) if $\mathcal{J}(f^+) \subset \mathcal{J}(g^+)$, $\mathcal{J}(f^-) \subset \mathcal{J}(g^-)$, and $\nu_f \ll \nu_g$ (in the sense of measures). f is said to be absolutely continuous if $f^+ \equiv f^- \equiv 0$, and $\nu_f \ll \lambda$; this is equivalent to saying that $f \ll g$, where g is the identity on a support of f .

If g is a fixed bv^* function, then every bv^* function f can be written uniquely in the form $f = f^{ac} + f^\perp$, where $f^{ac} \ll g$ and $f^\perp \perp g$; this follows easily from the corresponding fact for measures $[H_1]$, p. 134, Theorem C]. In the particular case when g is the identity on a support of f , the component f^{ac} has a well-known

explicit form [Sa, p. 119, Theorem 7.4]. Indeed, the derivative f' of f exists a.e.* and is integrable w.r.t. Lebesgue measure.

and for any real a we have

$$(8.4) \quad f^{ac}(t) = f^{ac}(a) + \int_a^t f'(s) ds.$$

We wish to obtain a similar explicit expression when g is not necessarily the identity. For this purpose, define a function

$f_{(g)} = df/dg$ on E^1 by

$$f_{(g)}(t) = \lim_{s \rightarrow t} \frac{f(s) - f(t)}{g(s) - g(t)}.$$

LEMMA 8.5. Let f and g in bv^* be continuous and nondecreasing. Then $f_{(g)}$ exists a.e. w.r.t. ν_g and is integrable w.r.t. ν_g over E^1 ; and for any real a we have

$$f^{ac}(t) = f^{ac}(a) + \int_a^t f_{(g)}(s) dg(s),$$

where $f = f^{ac} + f^\perp$ is the decomposition of f w.r.t. g .

The proof of this lemma proceeds by transforming the problem to one in which g is the identity on a support of f , applying (8.4), and then transforming back. It is not particularly difficult, but long and tedious. In order to avoid breaking the continuity of the

* "Almost everywhere"—i.e., everywhere except possibly in a set of measure 0. If the measure in question is μ , we will write "a.e. w.r.t. μ "; the only exception to this rule is when μ is Lebesgue measure, as in this case, when we simply write "a.e."

presentation in this section, we postpone the proof to an appendix.

If $f \in bv^*$ and r is real, we may define a function $\Delta_r f \in bv^*$ by

$$(\Delta_r f)(t) = f(t + r).$$

Note that for all r ,

$$f \text{ singular} \Rightarrow \Delta_r f \text{ singular};$$

$$f \text{ absolutely continuous} \Rightarrow \Delta_r f \text{ absolutely continuous};$$

$$f \perp g \Rightarrow \Delta_r f \perp \Delta_r g;$$

$$f \ll g \Rightarrow \Delta_r f \ll \Delta_r g.$$

LEMMA 8.6. Let $f, g \in bv^*$, let g be singular,
and let α and β be real numbers, $\alpha \neq \beta$. Then
 $\Delta_{r\alpha} f \perp \Delta_{r\beta} g$ for almost all r .

Proof. Start out by assuming that $\alpha = 1$, $\beta = 0$, and f and g are continuous and nondecreasing; these assumptions will be removed later.

For each r , let

$$(8.7) \quad \Delta_r f = f_r^{ac} + f_r^{\perp}$$

be the decomposition of $\Delta_r f$ w.r.t. g . Our first claim is that for each t ,

$$(8.8) \quad f_r^{ac}(t) \text{ is measurable in } r.$$

Indeed, let $[c, d]$ be a support of f ; then $[c-r, d-r]$ is a support of $\Delta_r f$,

and $f_r^{ac}(c-r) = (\Delta_r f)(c-r) = f(c)$. Hence from Lemma 8.5 it follows that

$$f_r^{ac}(t) = f(c) + \int_{c-r}^t \frac{d(\Delta_r f)}{dg}(s) dg(s).$$

Now it is easily verified that $(d(\Delta_r f)/dg)(s)$ is simultaneously measurable in r and s ; and then it follows that its integral w. r. t. $g(s)$ is measurable in r . This proves (8.8). Incidentally, it is only for this purpose that Lemma 8.5 is needed.

Let $\Gamma = [\gamma, \delta]$ be a closed bounded interval, and write

$$\begin{aligned} F(t) &= \int_{\Gamma} (\Delta_r f)(t) dr \\ (8.9) \quad F^{ac}(t) &= \int_{\Gamma} f_r^{ac}(t) dr \\ F^L(t) &= \int_{\Gamma} f_r^L(t) dr \end{aligned}$$

for all t . From (8.7) it follows that

$$F = F^{ac} + F^L.$$

Let

$$H(t) = \int_0^t f(t) dt.$$

Clearly H is absolutely continuous, and

$$F(t) = \int_{\gamma}^{\delta} f(t+r) dr = H(\delta+t) - H(\gamma+t).$$

Hence F , too, is absolutely continuous.

Let $v_r^{ac} = v_{f_r^{ac}}$, and $v^{ac} = v_{F^{ac}}$. If A is an interval of the form $[0, a]$, then from (8.9) it follows that

$$\nu^{ac}(A) = \int_{\Gamma} \nu_r^{ac}(A) dr.$$

Both sides of this equation are measures in A , so the equation must hold for all measurable A . From $f_r^{ac} \ll g$ it follows that $\nu_r^{ac} \ll \nu_g$. Hence if $U \subseteq E^1$ is such that $\nu_g(U) = 0$, then $\nu_r^{ac}(U) = 0$ for all r , and so

$$\nu^{ac}(U) = \int_{\Gamma} \nu_r^{ac}(U) dr = 0.$$

Since F is absolutely continuous and g singular, we have $\nu_F \perp \nu_g$, and so we may partition E^1 into disjoint sets U and V such that $\nu_g(U) = \nu_F(V) = 0$, so that in particular it follows that $\nu^{ac}(U) = 0$. Setting $\nu^\perp = \nu_{F^\perp}$, we deduce from $F = F^{ac} + F^\perp$ that

$$\nu_F = \nu^{ac} + \nu^\perp.$$

Now for any set $A \subseteq E^1$ we have

$$\begin{aligned} \nu^{ac}(A) + \nu^\perp(A) &= \nu_F(A) = \nu_F(A \cap U) + \nu_F(A \cap V) \\ &= \nu_F(A \cap U) = \nu^{ac}(A \cap U) + \nu^\perp(A \cap U) = \nu^\perp(A \cap U). \end{aligned}$$

On the other hand, since all measures involved are nonnegative, we have $\nu^\perp(A) \geq \nu^\perp(A \cap U)$; hence $\nu^{ac}(A) = 0$. Since A was chosen arbitrarily, it follows that ν^{ac} vanishes identically, and hence that F^{ac} vanishes identically. Hence f_r^{ac} vanishes identically for almost all r , as was to be proved. It remains only to remove the restrictions on α , β , f , and g .

First we get rid of the assumption that f and g are non-decreasing. Since f and g are in bv^* and are continuous, and g is singular, there are continuous nondecreasing f^1, f^2, g^1, g^2 , in bv^* such that g^1 and g^2 are singular and $f = f^1 - f^2$, $g = g^1 - g^2$. Then from what we have already proved it follows that for almost all r we have $\Delta_r f^1 \perp g^1$, $\Delta_r f^1 \perp g^2$, $\Delta_r f^2 \perp g^1$, $\Delta_r f^2 \perp g^2$; and then it follows from (8.2) that

$$\Delta_r f = \Delta_r f^1 - \Delta_r f^2 \perp g^1 - g^2 = g.$$

Next, we allow f and g to have discontinuities. Then we may write $f = f^1 + f^2$, $g = g^1 + g^2$, where $f^1, f^2, g^1, g^2 \in bv^*$, f^1 and g^1 are continuous, g^1 is singular, and f^2 and g^2 are jump functions. Then for almost all r we have $\Delta_r f^1 \perp g^1$; and for all r we have $\Delta_r f^1 \perp g^2$ and $f^2 \perp g^1$. It remains only to prove that $\Delta_r f^2 \perp g^2$ for almost all r . Now $\mathcal{J}(f^2) - \mathcal{J}(g^2)$ (algebraic difference!) is denumerable, so

$$r \notin \mathcal{J}(f^2) - \mathcal{J}(g^2)$$

for almost all r . But for all such r we have

$$\mathcal{J}(\Delta_r f^2) \cap \mathcal{J}(g^2) = \emptyset,$$

and this in turn implies $\Delta_r f^2 \perp g^2$. So $\Delta_r f^2 \perp g^2$ a.e., and applying (8.2), we deduce $\Delta_r f \perp g$ a.e.

Finally from $\alpha \neq \beta$ and what we have already proved it follows that $\Delta_{(\alpha-\beta)r+\beta r} f \perp \Delta_{\beta r} g$ for almost all r , and the proof of Lemma 8.6 is complete.

COROLLARY 8.10. Let $f, g \in bv^*$, g singular,
 α and β distinct real numbers. Then for almost all r ,
the functions $f((1-r)t + r\alpha)$ and $g((1-r)t + r\beta)$ of the argument
 t are mutually singular.

Proof. For real s , define an operator Γ_s on bv^* by

$$(\Gamma_s h)(t) = h(st) ;$$

then $h_1 \perp h_2$ implies $\Gamma_s h_1 \perp \Gamma_s h_2$ for all s . From Lemma 8.6 we obtain $\Delta_{r\alpha} f \perp \Delta_{r\beta} g$ for almost all r ; hence setting $s = 1 - r$ above, we obtain

$$\Gamma_{1-r} \Delta_{r\alpha} f \perp \Gamma_{1-r} \Delta_{r\beta} g$$

for almost all r . The latter two functions are precisely those appearing in the statement of the corollary, so the proof is complete.

If f is a function of bounded variation defined on a finite interval $[c, d]$, then f may be extended in a natural way to all of $(-\infty, \infty)$ so that the extended function f^* will be in bv^* . This is done by defining

$$f^*(s) = \begin{cases} f(c), & s \leq c \\ f(s), & c \leq s \leq d \\ f(d), & d \leq s \end{cases} .$$

The terminology introduced above for bv^* functions may then be naturally extended to functions of bounded variation on any interval;

thus we shall say that f is absolutely continuous, singular, etc., if and only if f^* is absolutely continuous, singular, etc. We shall also define $\|f\|$ by

$$\|f\| = \|f^*\|.$$

LEMMA 8.11. Let m be a positive integer,
and let ξ^1, \dots, ξ^m be m non-atomic vector measures
of the same dimension on I . For each ordered
partition $\mathcal{T} = \{T^1, \dots, T^m\}$ of I into Borel sets,
consider the matrix $(\xi^1(T^1), \dots, \xi^m(T^m))$. Then the
set of all such matrices as \mathcal{T} ranges over all ordered
partitions of I into m Borel sets is convex and compact.

Proof. This is Theorem 1 of [D-S]. It is a consequence of Lyapunov's theorem [L].

PROPOSITION 8.12. Let $g_1, \dots, g_\ell \in bv^1$
be singular, let ν_1, \dots, ν_ℓ be pairwise different non-
atomic probability measures, and let $u \in AC$ (see Sec. 5).
Then

$$\|u + g_1 \cdot \nu_1 + \dots + g_\ell \cdot \nu_\ell\| = \|u\| + \|g_1\| + \dots + \|g_\ell\|.$$

Proof. Let

$$v = u + g_1 \cdot \nu_1 + \dots + g_\ell \cdot \nu_\ell.$$

Fix $\epsilon > 0$, and let C be a chain

$$\emptyset = S^0 \subset S^1 \subset \dots \subset S^m = I$$

such that

$$\|u\|_{\Omega} > \|u\| - \epsilon.$$

Let ν_0 be a non-atomic probability measure such that $u \ll \nu_0$, and let R denote the range of the vector measure $\nu = (\nu_0, \nu_1, \dots, \nu_t)$.

For $i=1, \dots, m$, let

$$T^i = S^i \setminus S^{i-1};$$

the T^i are disjoint, and $\bigcup_{i=1}^m T^i = I$. Let $L = \{1, \dots, t\}$. We claim that there is a point y in R such that

$$(8.13) \quad \frac{y_p - \nu_p(S^i)}{\nu_p(T^{i+1})} \neq \frac{y_q - \nu_q(S^i)}{\nu_q(T^{i+1})}$$

for all $i = 0, \dots, m-1$ and all $p, q \in L$ such that $p \neq q$ and $\nu_p(T^{i+1}) \neq 0$, $\nu_q(T^{i+1}) \neq 0$. To prove this, note that if M is a linear submanifold of E^{t+1} , then since R is convex, either $M \supset R$ or the dimension of the intersection $M \cap R$ is smaller than the dimension of R . So if R is contained in a finite union $\bigcup_j M_j$ of linear manifolds, then it must be contained in one of the M_j , for otherwise the dimension of the intersection $R \cap \bigcup_j M_j$ would be smaller than the dimension of R .

Now for each appropriate p , q , and i , the equation

$$\frac{y_p - \nu_p(S^i)}{\nu_p(T^{i+1})} = \frac{y_q - \nu_q(S^i)}{\nu_q(T^{i+1})}$$

defines a linear manifold. So either there is a y in R satisfying none of these equations, in which case our claim is established, or all y in R satisfy the same one. But the second alternative is impossible, because such an equation, together with the existence of two y 's with $y_p = y_q$ (namely $y = v(\emptyset)$ and $y = v(I)$), implies $y_p = y_q$ for all y in R , contradicting the assumption that v_p and v_q are distinct. This proves (8.13).

Choose $T^0 \subset I$ so that

$$v(T^0) = y,$$

where y obeys (8.13); such a T^0 exists because y was chosen to be in the range R of v . Let $T^{m+1} = I \setminus T^0$. Define non-atomic vector measures ξ^0, \dots, ξ^{m+1} on I by

$$\xi^i(S) = v(S \cap T^i),$$

and apply Lemma 8.11 with $m + 2$ instead of m . For the partition $(\emptyset, T^1, \dots, T^m, \emptyset)$ we obtain the matrix $(0, v(T^1), \dots, v(T^m), 0)$; for the partition $(T^0, \emptyset, \dots, \emptyset, T^{m+1})$ we obtain the matrix $(y, 0, \dots, 0, e-y)$. Then since the set of all such matrices is convex, we may for each $r \in [0, 1]$ find a partition

$$(U_r^0, U_r^1, \dots, U_r^m, U_r^{m+1})$$

such that

$$\xi(U_r^0) = ry,$$

$$\xi(U_r^{m+1}) = r(e-y),$$

and

$$\xi(U_r^i) = (1-r) \vee(T^i)$$

for all other i . Setting

$$T_r^i = U_r^i \cap T^i,$$

and using the definition of ξ , we deduce

$$\vee(T_r^0) = ry,$$

$$\vee(T_r^{m+1}) = r(e-y),$$

and

$$\vee(T_r^i) = (1-r) \vee(T^i)$$

for all other i ; also the T_r^i are disjoint (for fixed r), and $T_r^i \subset T^i$.

Setting

$$S_r^i = T_r^0 \cup \dots \cup T_r^i$$

for $i = 0, \dots, m$, we obtain a chain

$$\emptyset \subset S_r^0 \subset S_r^1 \subset \dots \subset S_r^m \subset I$$

such that

$$(8.14) \quad \vee(S_r^i) = (1-r) \vee(S^i) + ry.$$

(see Fig. 3).

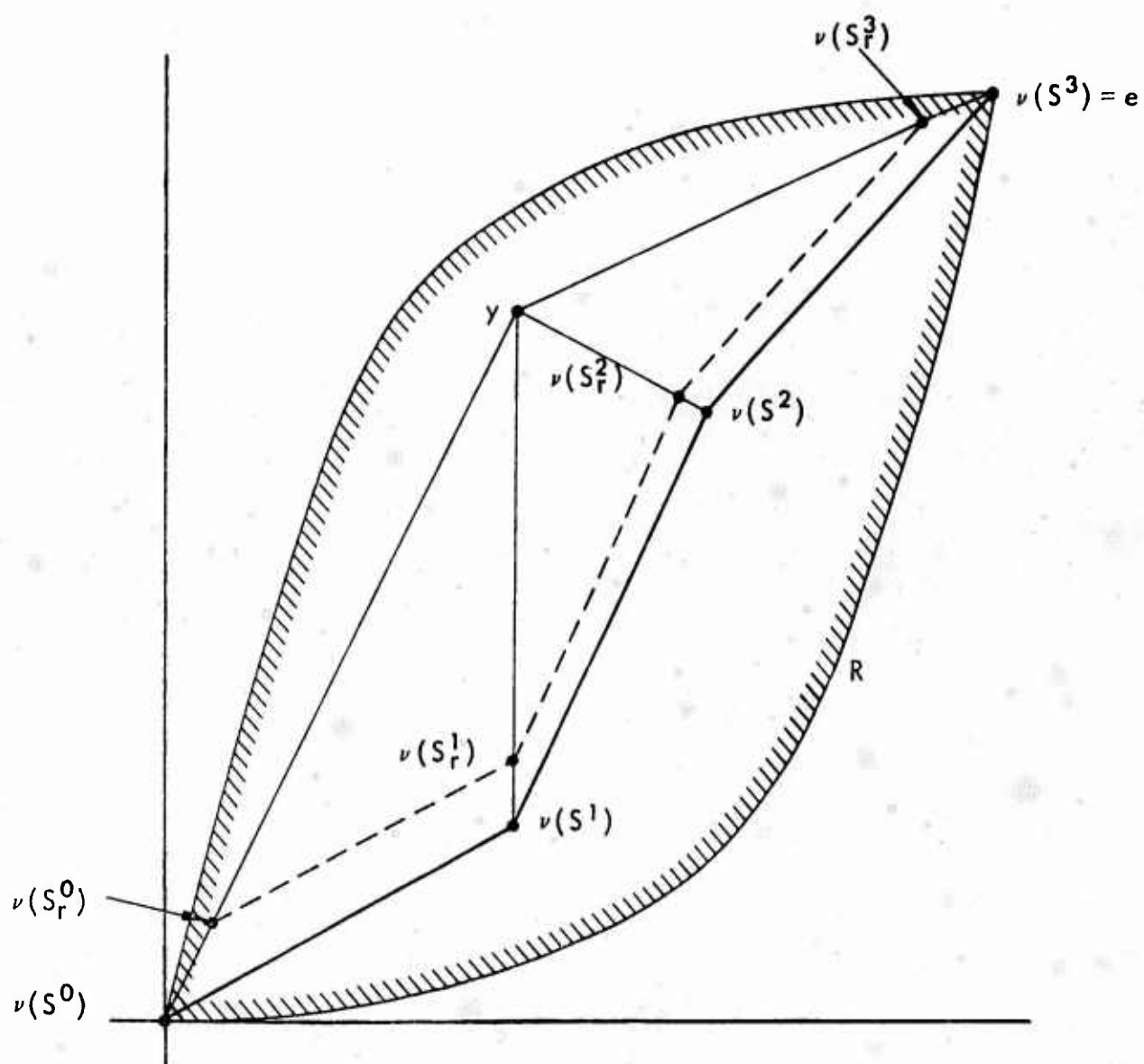


Figure 3

By Lemma 5.4, we can associate with each $\alpha \in [0, m]$ and each $r \in [0, 1]$ a set $S_r^\alpha \subset I$ such that $\alpha > \beta$ implies $S_r^\alpha \supset S_r^\beta$, such that for integer α , the S_r^α are the same as the S_r^i defined above, and such that for $i = 0, \dots, m-1$ and $t \in [0, 1]$,

$$(8.15) \quad v(S_r^{i+t}) = (1-t)v(S_r^i) + tv(S_r^{i+1}) = v(S_r^i) + tv(T_r^{i+1}).$$

Define functions $f^r, h^r, h_1^r, \dots, h_\ell^r$ by

$$(8.16) \quad \begin{aligned} f^r(\alpha) &= v(S_r^\alpha) \\ h^r(\alpha) &= u(S_r^\alpha) \\ h_p^r(\alpha) &= g_p(v_p(S_r^\alpha)) \end{aligned}$$

for all $\alpha \in [0, m]$ and $p \in L$. We claim that

$$(8.17) \quad \|v\| \geq \|f^r\| = \|h^r + \sum_{p \in L} h_p^r\|;$$

$$(8.18) \quad h^r \text{ is absolutely continuous.}$$

In fact, (8.17) is trivial, since $\|f^r\|$ is the sup of the variation of v over a particular family of chains. To prove (8.18), note that

$$v_0(S_r^{i+t}) = v_0(S_r^i) + tv_0(T_r^{i+1}).$$

From this and $u \ll v_0$ it follows that $h^r|_{[i, i+1]}$ is absolutely continuous* (the case $v_0(S_r^{i+1}) = v_0(S_r^i)$ needs separate treatment, but is trivial), and hence that h^r itself is absolutely continuous.

* Compare the second half of the proof of Theorem C.

For $p \in L$, $i = 0, \dots, m-1$, and all real t , set

$$g_p^i(t) = g_p^*(v_p(S^i) + tv_p(T^{i+1})).$$

Since the g_p are singular, so are the g_p^i . Setting, for $v_p(T^{i+1}) \neq 0$,

$$\alpha_p^i = \frac{y - v_p(S^i)}{v_p(T^{i+1})},$$

we obtain (by (8.14), (8.15), (8.16))

$$(8.19) \quad h_p^r(i+t) = g_p^i((1-r)t + r\alpha_p^i)$$

for $t \in [0, 1]$. If we fix i and apply (8.13) and Corollary 8.10, it follows that the functions on the right side of (8.19) are mutually singular for different p . Since mutually singular functions in bv^* remain so when restricted to a finite interval, we deduce that for almost all r ,

$$(8.20) \quad h_p^r|_{[i, i+1]} \perp h_q^r|_{[i, i+1]}$$

when $p \neq q$ and $v_p(T^{i+1}) \neq 0 \neq v_q(T^{i+1})$. When $v_p(T^{i+1}) = 0$, then by (8.15) and (8.16), we have

$$h_p^r|_{[i, i+1]} \equiv g_p(v_p(S_r^i));$$

hence $h_p^r|_{[i, i+1]}$ is a constant and hence mutually singular to all functions on $[i, i+1]$. Hence (8.20) holds whenever $p \neq q$. This holds for fixed i ; but since there are only finitely many i , it follows that for almost all r , (8.20) holds for all i and all $p \neq q$. Hence for almost all r

$$(8.21) \quad h_p^r \perp h_q^r$$

whenever $p \neq q$.

From (8.19) it follows that $h_p^r|_{[i, i+1]}$ is singular for all i , and hence h_p^r itself is singular. Since h^r is absolutely continuous (8.18), we deduce $h^r \perp h_p^r$ for all p . Then (8.17), (8.21), and Remark 8.3 yield

$$(8.22) \quad \|v\| \geq \|h^r + \sum_{p \in L} h_p^r\| \geq \|h^r\| + \sum_{p \in L} \|h_p^r\|$$

for almost all r .

Equations (8.14), (8.15), and (8.16) yield

$$\|h_p^r\| = \|g_p|_{[ry_p, 1 - r(1 - y_p)]}\|.$$

Since g_p is continuous at 0 and at 1 (here is the only use of this assumption in this proposition), it follows that $\|h_p^r\| \rightarrow \|g_p\|$ as $r \rightarrow 0$; in particular,

$$(8.23) \quad \sum_{p \in L} \|h_p^r\| \geq \sum_{p \in L} \|g_p\| - \epsilon$$

for sufficiently small r .

Finally, we have

$$S_r^i \setminus T_r^0 = \bigcup_{j=1}^i T_r^j \subset \bigcup_{j=1}^i T^j = S^i$$

and

$$|\nu_0(S^i) - \nu_0(S_r^i \setminus T_r^0)| = r\nu_0(S^i);$$

also $S_r^i \setminus T_r^0 \subset S_r^i$, and

$$|\nu_0(S_r^i) - \nu_0(S_r^i \setminus T_r^0)| = ry.$$

From this and $u \ll \nu_0$ it follows that $u(S_r^i) \rightarrow u(S^i)$ as $r \rightarrow 0$. On the other hand, clearly

$$\|h^r\| \geq \sum_{i=1}^m |u(S_r^i) - u(S_r^{i-1})|;$$

letting $r \rightarrow 0$, we obtain that for sufficiently small r ,

$$\|h^r\| \geq \sum_{i=1}^m |u(S^i) - u(S^{i-1})| - \epsilon = \|u\|_0 - \epsilon \geq \|u\| - 2\epsilon.$$

Combining this with (8.22) and (8.23), and choosing a small r appropriately, we deduce

$$\|v\| \geq \|u\| + \sum_{p \in L} \|g_p\| - 3\epsilon.$$

Since r does not appear in this inequality, it holds for all ϵ , and so

$$\|v\| \geq \|u\| + \sum_p \|g_p\|.$$

The opposite inequality is trivial, and so Proposition 8.12 is proved.

* * * * *

Let $s'NA$ be the subspace of BV spanned by all scalar measure functions $f \bullet \mu$, where f in bv' is singular and $\mu \in NA$ is a probability measure. From Proposition 8.12 it follows that

$$(8.24) \quad u \in AC, w \in s'NA \Rightarrow \|u + w\| = \|u\| + \|w\|.$$

From Theorem C and the fact that every bv' function can be decomposed into an absolutely continuous and a singular component in bv' , it follows that

$$(8.25) \quad bv'NA = pNA + s'NA .$$

We wish to show that this sum is direct, i. e., that $pNA \cap s'NA = \{0\}$.

In fact, we have the following more general result:

$$(8.26) \quad AC \cap s'NA = \{0\} .$$

To prove this, suppose $v \in AC \cap s'NA$; set $u = v$, $w = -v$, apply (8.24), and deduce

$$0 = \|v - v\| = \|v\| + \|-v\| = 2\|v\| .$$

Hence $v = 0$, as was to be proved.

From (8.25), (8.26), and Corollary 5.3 it follows that

$$AC \cap bv'NA = pNA .$$

Hence the set function $v \in AC \setminus pNA$ of Example 5.8 is not in $bv'NA$ either.

We are now ready for the

Proof of Theorems A and B. First, we prove the existence of a value on $s'NA$. Let Q be the set of set functions v of the form

$$(8.27) \quad v = g_1 \cdot v_1 + \dots + g_l \cdot v_l ,$$

where the g_p are singular members of bv' and the ν_p are probability measures in NA. For each v in Q and each representation \mathcal{R} of v in the form (8.27), we may define a measure $\theta = \theta_{v, \mathcal{R}}$ by

$$(8.28) \quad \theta = \sum_{p=1}^l g_p(1) \nu_p.$$

Then we claim that

$$(8.29) \quad \|\theta\| \leq \|v\|.$$

To prove this, assume first that the ν_p are pairwise different.

Then from Proposition 8.12 and the fact that the ν_p are probability measures it follows that

$$\|\theta\| \leq \sum_p |g_p(1)| \|\nu_p\| = \sum_p |g_p(1)| \leq \sum_p \|g_p\| = \|v\|,$$

establishing (8.29). If the ν_p are not necessarily pairwise different, we may group terms (e. g., if $\nu_1 = \nu_2$ we may write $(g_1 + g_2) \circ \nu_1$ instead of $g_1 \circ \nu_1 + g_2 \circ \nu_2$). Although this leads to a different representation for v , it is easily seen that it does not change θ ; hence since (8.29) holds for the "grouped" representation, it also holds for the original one.

Now let

$$v = \sum_{p=1}^k g'_p \circ \nu'_p$$

be a different representation of v in the form (8.27), which we denote \mathcal{R}' . Then $0 = v - v$ has a representation \mathcal{P} given by

$$0 = \sum_{p=1}^l g_p \cdot v_p - \sum_{p=1}^k g'_p \cdot v'_p.$$

From (8.29) we obtain

$$\|\theta_{0, \rho}\| \leq \|0\| = 0,$$

so $\theta_{0, \rho} = 0$. On the hand it is easily verified that

$$\theta_{0, \rho} = \theta_{v, \mathcal{R}} - \theta_{v, \mathcal{R}'};$$

hence

$$\theta_{v, \mathcal{R}} = \theta_{v, \mathcal{R}'}.$$

Thus $\theta_{v, \mathcal{R}}$ depends on the set function v only, and not on its representation \mathcal{R} ; so we may define

$$\varphi'v = \theta_{v, \mathcal{R}}$$

for an arbitrary representation \mathcal{R} .

Clearly Q is dense in $s'NA$. We have already defined φ' on Q ; it is easily verified that Q is a linear subspace of $s'NA$, and that φ' is linear on Q . From (8.29) it follows that $\|\varphi'v\| \leq \|v\|$ for $v \in Q$, so φ' is continuous on Q and $\|\varphi'\| \leq 1$. If f is a normalized singular pv function and $v = f \cdot \lambda$, then $\varphi v = \lambda$, so $\|\varphi'v\| = 1 = \|v\|$, and hence $\|\varphi'\| \geq 1$. Therefore $\|\varphi'\| = 1$.

From the completeness of FA it then follows that φ' can be uniquely extended to be a continuous linear operator from $s'NA$ to FA , and this extension will also have norm = 1. That the extension

is indeed a value on $s'NA$ follows easily from (8.28) and Proposition 6.4.

Proposition 7.6 asserts the existence of a value φ_1 on pNA with $\|\varphi_1\| = 1$, and we have just proved that there is a value φ_2 on $s'NA$ with $\|\varphi_2\| = 1$. Note that by (8.25) and (8.26), each v in $bv'NA$ can be uniquely decomposed into a $u \in pNA$ and a $w \in s'NA$ such that

$$v = u + w.$$

We then define φ on $bv'NA$ by

$$\varphi v = \varphi_1 u + \varphi_2 w.$$

This φ is clearly linear; moreover from (8.24) it follows that

$$\|\varphi v\| = \|\varphi_1 u + \varphi_2 w\| \leq \|\varphi_1 u\| + \|\varphi_2 w\| \leq \|u\| + \|w\| = \|u+w\| = \|v\|.$$

Hence $\|\varphi\| \leq 1$, and as we have already shown that $\|\varphi\| \geq 1$, it follows from Proposition 6.4 (with F the union of s' and the positive integer powers) that φ is a value on $bv'NA$. That the range of φ is NA follows from Propositions 6.1 and 4.4.

To prove the uniqueness, let A be the set of all $f \cdot \mu$, where $f \in s'$ and μ is a non-atomic probability measure. Clearly A is reproducing. Furthermore, for every $g \in s'$ there are monotonic g^+ and g^- in s' such that

$$f = g^+ - g^-$$

and

$$\|g\| = g^+(1) + g^-(1) = \|g^+\| + \|g^-\|.$$

From this and Proposition 8.12, applied with $u = 0$, we deduce that

$$(8.30) \quad \|v\| = \|v\|_A$$

for all $v \in A$. Hence by Proposition 4.12, if φ is a value on $s'NA = \bar{A}$, then φ is continuous. On the other hand, Proposition 6.1 determines φ on a spanning subset of $s'NA$; so by the continuity, which we have just proved, φ is determined on all of $s'NA$.

Now if φ is a value on $bv'NA$, then $\varphi|_{pNA}$ is a value on pNA and $\varphi|_{s'NA}$ is a value on $s'NA$. The former is determined by Proposition 7.11, and the latter by what we have just proved; this completes the proof of Theorem A. Theorem B follows at once from Theorem A and from Propositions 7.1 and 7.6.

In Sec. 3 we asserted that if in the definition of value we replace positivity by continuity, then Theorem A remains true as it stands. This assertion follows easily from the above line of proof.

From (8.30) and Proposition 4.12 it follows that $s'NA$ is reproducing and that

$$\|v\| = \|v\|_{s'NA}$$

for all $v \in s'NA$. Combining this with Proposition 7.25 and formula (8.24) we deduce

PROPOSITION 8.31. $bv'NA$ is reproducing, and

$$\|v\| = \|v\|_{bv'NA}$$

for all $v \in bv'NA$.

Finally, we wish to establish formula (3.1) under the condition (3.3), which is weaker than the condition as stated in Theorem B. To this end, we first prove

PROPOSITION 8.32. Let $v \in bv'NA$ be such that there is a positive integer n , an n -dimensional vector μ of non-atomic measures, and a convex neighborhood U in E^n of the diagonal $[0, \mu(I)]$ such that

$$\mu(S) \in U \Rightarrow v(S) = 0.$$

Then $v \in pNA$.

Proof. Let $v = u + w$ be the decomposition, in accordance with (8.25), of v into set functions $u \in pNA$ and $w \in s'NA$.

For each $\epsilon > 0$ we may find a set function

$$w^\epsilon = g_1 \cdot v_1 + \dots + g_\ell \cdot v_\ell,$$

where the $g_p \in bv'$ are singular and the v_p are probability measures in NA , such that

$$\|w - w^\epsilon\| < \epsilon.$$

Let $v_0 \in NA$ be such that $u \ll v_0$. We now imitate the proof of Proposition 8.12, applying it to the $(n + \ell + 1)$ -dimensional vector measure $\xi = (\mu, v_0, v_1, \dots, v_\ell)$. Let $T \subset I$ be such that $v_p(T) \neq v_q(T)$ whenever $p, q > 0$ are different; the existence of such a T is a special case of (8.13), and it may also be verified directly. Now let $r \in [0, 1]$. By applying Lyapunov's theorem, first to T and then to $I \setminus T$, we may find sets S^0 and S^1 such that $S^0 \subset T \subset S^1 \subset I$ and

$$\xi(S^0) = r\xi(T)$$

$$\xi(S^1) = (1-r)\xi(I) + r\xi(T)$$

(see Fig. 4). For small r , therefore, we will have $\xi(S^0)$ close to 0 and $\xi(S^1)$ close to $\xi(I)$; hence $\mu(S^0)$ is close to 0, $\mu(S^1)$ is close to $\mu(I)$, and in particular we may choose r sufficiently small so that $\mu(S^0)$ and $\mu(S^1)$ are in U . Next, we apply Lemma 5.4 to assign to each $t \in [0, 1]$ a set S^t such that S^0 and S^1 are as defined, $t > s$ implies $S^t \supset S^s$, and

$$\xi(S^t) = t\xi(S^1) + (1-t)\xi(S^0).$$

It then follows that all the $\mu(S^t)$ are convex combinations of $\mu(S^0)$ and $\mu(S^1)$, and hence they are in U . Now define functions in bv^* with support $[0, 1]$ as follows (for $t \in [0, 1]$):

$$\tilde{w}(t) = w(S^t)$$

$$\tilde{w}^\epsilon(t) = w^\epsilon(S^t)$$

$$\tilde{g}_p(t) = g_p(v_p(S^t)) = g_p((1-r)t + rv_p(t))$$

$$\tilde{u}(t) = u(S^t).$$

Then

$$\|\tilde{w} - \tilde{w}^\epsilon\| \leq \|w - w^\epsilon\| < \epsilon$$

and for $t \in [0, 1]$,

$$\tilde{u}(t) + \tilde{w}(t) = h(\mu(S^t)) = 0,$$

since $\mu(S^t) \in U$. Now from $u \ll v_0$ and $v_0(S^t) = (1-r)t v_0(I) + r v_0(T)$,

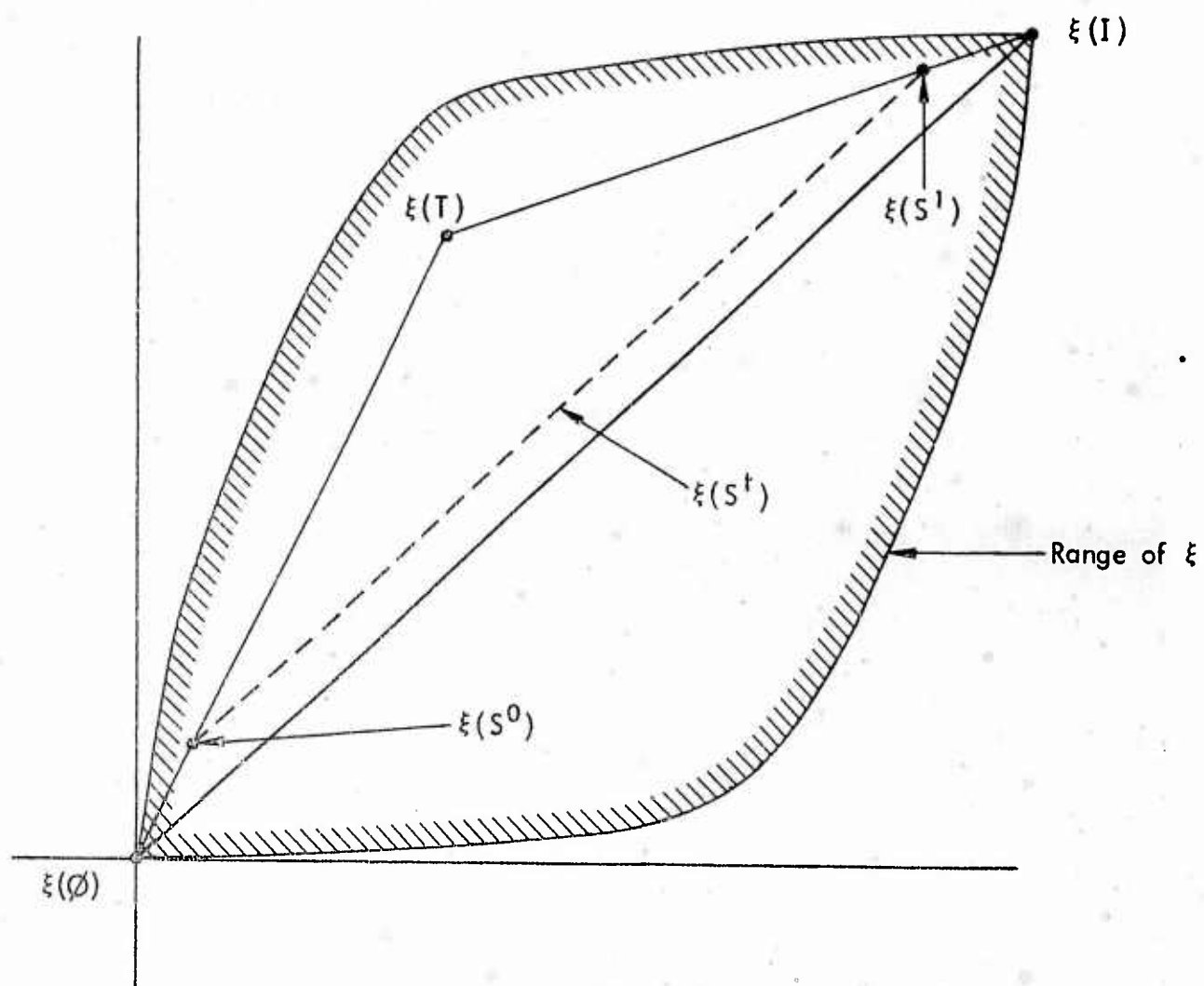


Figure 4

it follows that \tilde{u} is absolutely continuous, and hence $\tilde{u} \perp \tilde{g}_i$ for all i , since the \tilde{g}_i are singular. Next, we note that there are only denumerably many r such that for some i , g_i has a jump at $rv_i(T)$ or at $(1-r) + rv_i(T)$. Hence by Corollary 8.10, we may find arbitrarily small r such that $\tilde{g}_i \perp \tilde{g}_j$ for all i and j . Choosing such an r and applying Remark 8.3, we find

$$\begin{aligned} 0 = \|\tilde{u} + \tilde{w}\| &= \|\tilde{u} + \tilde{w}^\epsilon + \tilde{w} - \tilde{w}^\epsilon\| \geq \|\tilde{u} + \sum_{p=1}^l \tilde{g}_p\| - \|\tilde{w} - \tilde{w}^\epsilon\| \\ &> \|\tilde{u}\| + \sum_{p=1}^l \|\tilde{g}_p\| - \epsilon. \end{aligned}$$

In particular,

$$\sum_{p=1}^l \|\tilde{g}_p\| < \epsilon.$$

Now since $g_p \in bv'$, it follows that $\|\tilde{g}_p\| \rightarrow \|g_p\|$ as $r \rightarrow 0$. Hence letting $r \rightarrow 0$, we obtain

$$\sum_{p=1}^l \|g_p\| \leq \epsilon.$$

Hence

$$\begin{aligned} \|w\| &\leq \sum_{p=1}^l \|g_p \cdot v_p\| + \|w - w^\epsilon\| \\ &= \sum_{p=1}^l \|g_p\| + \|w - w^\epsilon\| < 2\epsilon. \end{aligned}$$

Letting $\epsilon \rightarrow 0$, we obtain $w = 0$. Hence $v = u \in pNA$, and the proof of Proposition 8.32 is complete.

Now assume condition (3.3); that is, that $f \cdot \mu \in bv'NA$ is such that μ is an n -dimensional vector of measures in NA with range R

and f is continuously differentiable on $U \cap R$, where U is a convex neighborhood in E^n of the diagonal $[0, \mu(I)]$. Let f^* be a function that is continuously differentiable on all of R and coincides with f on $U \cap R$ (the existence of f^* is well-known; see [W]). Let $v = f \circ \mu - f^* \circ \mu$; then by Proposition 8.32, $v \in pNA$. But by Proposition 7.1, $f^* \circ \mu \in pNA$ as well; hence

$$f \circ \mu = v + f^* \circ \mu \in pNA.$$

Now apply Proposition 7.6, setting the v of that proposition equal to $f \circ \mu$ here, setting the f of that proposition equal to f^* here, and setting the U of that proposition equal to $U \cap R$ here (μ remains unchanged). Then since f and f^* are equal near the diagonal, we obtain (3.1). This completes the proof of the stronger form of Theorem B.

It is tempting to try to simplify this section by defining the notion of "orthogonality" in BV by: $v \perp w$ if and only if $\|\alpha v + \beta w\| = |\alpha| \|v\| + |\beta| \|w\|$ for all real α and β (this of course is different from the ordinary notion of orthogonality in, say, a Hilbert space). If one could then prove that

$$(8.33) \quad u \perp w \text{ and } v \perp w \text{ imply } u + v \perp w,$$

then the entire treatment would be greatly shortened and simplified. This notion of orthogonality can be defined in any Banach space, and one might have hoped that (8.33) is always true. Unfortunately, this is not the case. Consider, for example, the 3-dimensional space (with norm $\|\cdot\|$) in which

the unit ball is the convex hull of the 8 points $(\pm \frac{1}{2}, \pm \frac{1}{2}, \pm \frac{1}{2})$ and the 3 unit vectors e_1, e_2, e_3 and their negatives $-e_1, -e_2, -e_3$. Then $e_1 \perp e_2$ and $e_1 \perp e_3$ but $\|e_1 + e_2 + e_3\|' = 2 < \|e_1\|' + \|e_2 + e_3\|'$. As for BV, this does not enjoy property (8.33) either, but we will not quote the example here.

9. A VARIATION ON A THEME OF THEOREM B

In this section we prove a theorem that relaxes somewhat the differentiability conditions on f under which it may be concluded that $f \circ \mu \in \text{pNA}$. In particular, consider the case in which μ is a vector of measures in NA^+ , f is defined on the nonnegative orthant of E^n , and for each i , the partial derivative $f_i(x) = \partial f / \partial x_i$ exists and is continuous whenever it is defined as a two-sided derivative, i. e., whenever $x_i > 0$. Thus f is continuously differentiable in the interior of the orthant; and on each of the faces (of various dimensions) of the orthant, some of the partial derivatives, but not all, must exist and be continuous. At the origin none of the partial derivatives need exist.

Under these conditions it can still be proved that $f \circ \mu$ is in pNA , if it is also assumed that f is increasing in each of its variables. The precise statement is Proposition 9.17 below.

Lemma 7.4 is useless in this context. Instead of working with the C^1 norm $\| \cdot \|_1$, we define and work with a variation norm for functions of several variables. Because the range of μ is necessarily compact, it is natural to work with the cube rather than with the entire orthant.

At the end of the section we will briefly discuss a possible generalization from the orthant to arbitrary convex closed sets in E^n containing the origin.

For x and y in E^n we will write $x \geq y$ if $x_i \geq y_i$ for all i . A real function f of n real variables is said to be nondecreasing if $x \geq y$ implies $f(x) \geq f(y)$ for all x and y for which f is defined.

Suppose f is a nondecreasing continuous real function on the unit square $[0, 1]^2$. Let m be even, and consider a "staircase" sequence of points $x^0 \leq \dots \leq x^m$ in the square, i.e., a sequence in which x^{2j+1} differs from x^{2j} , if at all, only in the first coordinate; and x^{2j} differs from x^{2j-1} , if at all, only in the second coordinate. Then the total increment $\Delta = f(x^m) - f(x^0)$ of the function f over the sequence can be split into two parts: the increment

$$\Delta_1 = \sum_j (f(x^{2j+1}) - f(x^{2j}))$$

over the horizontal segments, and the increment

$$\Delta_2 = \sum_j (f(x^{2j}) - f(x^{2j-1}))$$

over the vertical segments.

For $\delta \geq 0$, let $\Delta_1(\delta)$ denote the supremum of Δ_1 over all staircase sequences involving only points x for which $x_1 \leq \delta$ (see Fig. 5). Clearly if $\delta = 0$ the horizontal segments all disappear, and we necessarily have $\Delta_1(0) = 0$. Because of the continuity of f , it is reasonable to conjecture that

$$(9.1) \quad \Delta_1(\delta) \rightarrow 0 \text{ as } \delta \rightarrow 0.$$

But in fact, under the conditions we have stated, (9.1) is false!

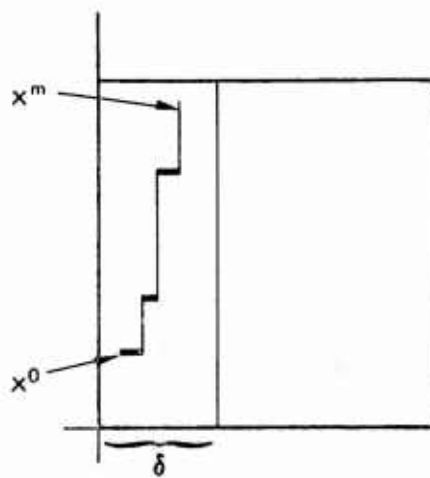


Figure 5

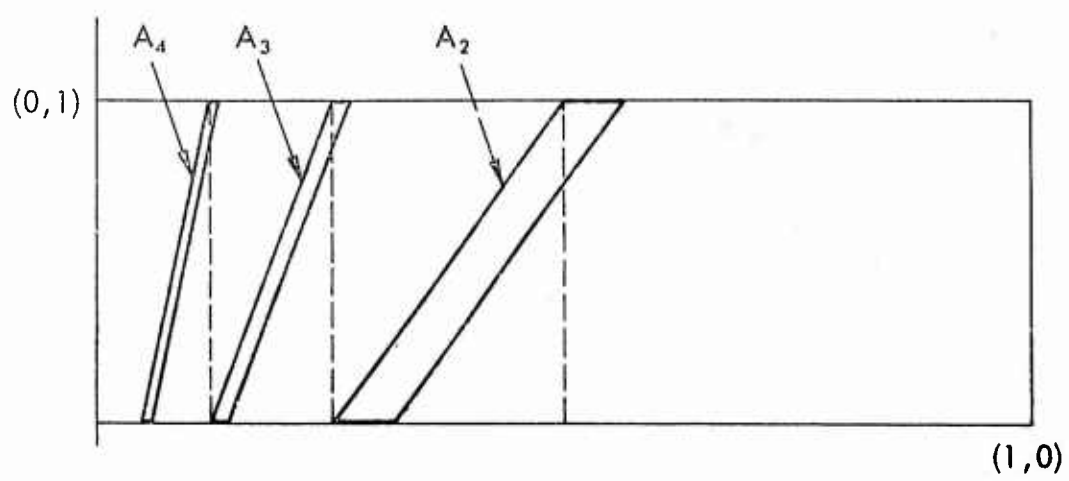


Figure 6

Indeed, for each $k \geq 2$ let $A_k \subset [0, 1]^2$ be the parallelogram whose vertices are $(2^{-k}, 0)$, $(2^{-k} + 4^{-k}, 0)$, $(2^{-k+1} + 4^{-k}, 1)$, and $(2^{-k+1}, 1)$ (see Fig. 6). Then we may find a nondecreasing continuous function f on the square such that for $x \in A_k$,

$$(9.2) \quad f(x) = f(x_1, x_2) = 2^k x_1 + 2^{-k+1} - 1.$$

For example, for x between A_k and A_{k-1} we may define

$$f(x) = 2^{-k+1} + x_2 + (x_1 - 2 \cdot 4^{-k}) / (1 - 2^{-k} + x_2);$$

for x to the right of A_2 we may define f by the same formula that defines f on A_2 , i.e., $f(x) = 4x_1 - \frac{1}{2}$; and for $x_1 = 0$ we may define $f(x) = x_2$. The monotonicity and continuity of f are easily verified.

If

$$(2^{-k}, 0) = x^0 \leq \dots \leq x^m = (2^{-k+1}, 1)$$

is a staircase sequence all of whose points are in A_k , then from (9.2) it follows at once that the vertical increment Δ_2 vanishes, so the total increment Δ equals the horizontal increment Δ_1 .

But $\Delta = f(2^{-k+1} + 4^{-k}, 1) - f(2^{-k}, 0) = 1 + 2^{-k}$. As $k \rightarrow \infty$, we have $x_1^m = 2^{-k+1} + 4^{-k} \rightarrow 0$, but Δ_1 tends to 1, not 0.

Note that the example can easily be modified so that f is continuously differentiable whenever $x_1 > 0$ and $x_2 > 0$.

In order to prove (9.1), it is of course sufficient to assume that f satisfies a uniform Lipschitz condition in x_1 , or in particular that the horizontal derivative $\partial f / \partial x_1$ exists and is

continuous on the entire square. But this is not necessary; rather surprisingly, it is sufficient to assume the existence and continuity of the vertical derivative $\partial f / \partial x_2$ for $x_2 > 0$. We first sketch the proof intuitively. Because of the continuity of the vertical derivative, the vertical increment of a sequence $\{x^j\}$ with $x_1^m \leq \delta$, δ small, can be approximated by the vertical increment of a nearby sequence $\{y^j\}$ with all $y_1^j = 0$. But for the latter, the vertical increment equals the total increment, and the total increment is clearly continuous (as a function of the endpoints of the sequence). So for small δ , the vertical increment Δ_2 must be close to the total increment $\Delta = \Delta_1 + \Delta_2$; hence Δ_1 is necessarily small. We now give the complete proof, for arbitrary n (rather than $n = 2$).

PROPOSITION 9.3. Let f be a continuous non-
decreasing real function on the unit cube $[0, 1]^n$ such
that for each i , the partial derivative $f_i = \partial f / \partial x_i$ exists*
and is continuous whenever $x_i > 0$. Then for every
 $\epsilon > 0$ there is a $\delta > 0$ such that for all i in $\{1, \dots, n\}$,
the following implication holds:

If $x^0 \leq \dots \leq x^m$ is a sequence of points in
 $[0, 1]^n$ such that for each j , x^j differs from x^{j-1}
in at most one coordinate, and such that $x_i^m < \delta$,

* When $x_i = 1$ the derivative is one-sided.

then

$$\sum_{j \in M(i)} (f(x^j) - f(x^{j-1})) < \epsilon ,$$

where $M(i)$ is the set of j in $\{1, \dots, m\}$ for which
 $x_i^j > x_i^{j-1}$.

Proof. The proof is by induction on the dimension n . Let ρ_n denote the proposition as stated. ρ_1 follows at once from the continuity of f . Assume ρ_{n-1} ; we wish to prove ρ_n . Let $N = \{1, \dots, n\}$ and $M = \{1, \dots, m\}$.

In proving ρ_n , we may restrict attention to a single fixed i ; for if we have found $\delta(i)$ for each i , we may define $\delta = \min_i \delta(i)$. Without loss of generality, we may let this fixed i be n . We will make use of the first $n-1$ derivatives only. Define a function f^* on $[0, 1]^{n-1}$ by

$$f^*(x_1, \dots, x_{n-1}) = f(x_1, \dots, x_{n-1}, 0).$$

f^* satisfies the hypotheses of ρ_{n-1} ; so for a given $\epsilon > 0$, we may choose a $\delta_1 > 0$ that corresponds to f^* and to $\epsilon/3n$ in accordance with the conclusions of ρ_{n-1} . Next, choose $\delta_2 > 0$ so that for all $r \in \{1, \dots, n-1\}$ and all x and y in $[0, 1]^n$, if $\|x - y\| \leq \delta_2$ and $x_r \geq \delta_1/2, y_r \geq \delta_1/2$, then

$$|f_r(x) - f_r(y)| < \epsilon/3n .$$

This is possible because of the uniform continuity of f_r in $\{x \in [0, 1]^n : x_r \geq \delta_1/2\}$.

Let δ_3 be such that $\|x - y\| \leq \delta_3$ implies $|f(x) - f(y)| < \epsilon/3$; this is possible because of the uniform continuity of f in $[0, 1]^n$.

Define

$$\delta = \delta(n) = \min(\delta_3, \delta_2).$$

Let x^0, \dots, x^m be given as in the statement of the proposition.

Without loss of generality, assume that

$$(9.4) \quad \|x^j - x^{j-1}\| < \min(\delta_1/2, \delta_2)$$

for all j ; if this is not already the case, we may insert additional points x^j so that it becomes true, without effecting any change in the statements appearing in the proposition.

For each $r \in N$, let $M(r) = \{j \in M: x_r^j > x_r^{j-1}\}$ and let $M_0 = \{j \in M: x^j = x^{j-1}\}$. The $M(r)$ are disjoint, because x^j can differ from x^{j-1} in at most one coordinate. For $r \in \{1, \dots, n-1\}$, let

$$K(r) = \{j \in M(r): x_r^j < \delta_1\}$$

$$L(r) = \{j \in M(r): x_r^j \geq \delta_1\} = M(r) \setminus K(r).$$

Define a sequence y^0, \dots, y^m by

$$y^j = (x_1^j, \dots, x_{n-1}^j, 0).$$

Now

$$\begin{aligned}
 f(y^m) - f(y^0) &= \sum_{j=1}^m (f(y^j) - f(y^{j-1})) \\
 &= \sum_{r=1}^n \sum_{j \in M(r)} (f(y^j) - f(y^{j-1})) \\
 &= \sum_{r=1}^{n-1} [\sum_{j \in K(r)} + \sum_{j \in L(r)}] (f(y^j) - f(y^{j-1})).
 \end{aligned}$$

The terms with $j \in M_0$ contribute nothing because then $x^j = x^{j-1}$, hence $y^j = y^{j-1}$; and when $j \in M(n)$ then $y^j = y^{j-1}$, so the terms with $r = n$ contribute nothing either.

For $r < n$, it follows from the definitions of $K(r)$ and δ_1 that

$$\sum_{j \in K(r)} (f(y^j) - f(y^{j-1})) < \epsilon / 3n.$$

Hence

$$\sum_{r=1}^{n-1} \sum_{j \in K(r)} (f(y^j) - f(y^{j-1})) < \epsilon / 3.$$

It follows that

$$(9.5) \quad \sum_{r=1}^{n-1} \sum_{j \in L(r)} (f(y^j) - f(y^{j-1})) > f(y^m) - f(y^0) - \epsilon / 3.$$

Let e^r be the r 'th unit vector $(0, \dots, 0, 1, 0, \dots, 0)$. Then for $r < n$ and $j \in L(r)$ the mean value theorem yields

$$f(y^j) - f(y^{j-1}) = f_r(y^j - \theta_r^j \Delta_r^j e^r) \Delta_r^j$$

where $0 \leq \theta_r^j \leq 1$ and $\Delta_r^j = y_r^j - y_r^{j-1}$. Similarly

$$f(x^j) - f(x^{j-1}) = f_r(x^j - \psi_r^j \Delta_r^j e^r) \Delta_r^j,$$

where $0 \leq \psi_r^j \leq 1$. Now since $j \in L(r)$, we have $x_r^j \geq \delta_1$; from (9.4) it then follows that $x_r^j - \Delta_r^j > \delta_1/2$. Hence

$$(9.6) \quad \left\{ \begin{array}{l} (x^j - \psi_r^j \Delta_r^j e^r)_r \geq x_r^j - \Delta_r^j > \delta_1/2 \\ \text{and} \\ (y^j - \theta_r^j \Delta_r^j e^r)_r \geq y_r^j - \Delta_r^j = x_r^j - \Delta_r^j > \delta_1/2. \end{array} \right.$$

Furthermore, we have

$$\begin{aligned} \|(x^j - \psi_r^j \Delta_r^j e^r) - (y^j - \theta_r^j \Delta_r^j e^r)\| &= \|x_n^j e_n + (\theta_r^j - \psi_r^j) \Delta_r^j e^r\| \\ &\leq \max(x_n^j, \Delta_r^j) \leq \delta_2; \end{aligned}$$

indeed $\Delta_r^j \leq \delta_2$ because of (9.4), and $x_n^j \leq x_n^m < \delta \leq \delta_2$. Combining this with (9.6) and the definition of δ_2 , we deduce

$$\begin{aligned} &|[f(y^j) - f(y^{j-1})] - [f(x^j) - f(x^{j-1})]| \\ &= |f_r(y^j - \theta_r^j \Delta_r^j e^r) - f_r(x^j - \psi_r^j \Delta_r^j e^r)| \Delta_r^j < \frac{\epsilon}{3n} \Delta_r^j. \end{aligned}$$

From this it follows that

$$\begin{aligned} f(x^m) - f(x^0) &= \sum_{j=1}^m (f(x^j) - f(x^{j-1})) \\ &= \sum_{r=1}^n \sum_{j \in M(r)} (f(x^j) - f(x^{j-1})) \\ &\geq \sum_{r=1}^{n-1} \sum_{j \in L(r)} (f(x^j) - f(x^{j-1})) + \sum_{j \in M(n)} (f(x^j) - f(x^{j-1})) \\ &\geq \sum_{r=1}^{n-1} \sum_{j \in L(r)} (f(y^j) - f(y^{j-1})) - \frac{\epsilon}{3n} \sum_{r=1}^{n-1} \sum_{j \in L(r)} \Delta_r^j \\ &\quad + \sum_{j \in M(n)} (f(x^j) - f(x^{j-1})). \end{aligned}$$

Now $\sum_{j \in L(r)} \Delta_r^j \leq \sum_{j \in M(r)} \Delta_r^j = x_r^m - x_r^0 \leq 1$; applying this and (9.5)

to the above string of inequalities, we deduce

$$f(x^m) - f(x^0) > f(y^m) - f(y^0) - \frac{\epsilon}{3} - \frac{\epsilon}{3} + \sum_{j \in M(n)} (f(x^j) - f(x^{j-1})).$$

Since $x^0 \geq y^0$, it follows that

$$\begin{aligned} \sum_{j \in M(n)} (f(x^j) - f(x^{j-1})) &< f(x^m) - f(y^m) - (f(x^0) - f(y^0)) + 2\epsilon/3 \\ &< f(x^m) - f(y^m) + 2\epsilon/3. \end{aligned}$$

Now $\|x^m - y^m\| = x_n^m < \delta$; since $\delta \leq \delta_3$, it follows that

$f(x^m) - f(y^m) < \epsilon/3$, and ρ_n follows. This completes the proof of

Proposition 9.3.

If f is a real function defined on the unit cube $[0, 1]^n$, with $f(0) = 0$, then the variation $\|f\|$ of f is defined to be

$$\sup \sum_{i=1}^k |f(x^{i-1}) - f(x^i)|,$$

where the sup is taken over all finite sequences $\{x^0, \dots, x^k\}$ such

that $0 = x^0 \leq x^1 \leq \dots \leq x^k = (1, \dots, 1)$.^{*} The function f is said

to be of bounded variation if $\|f\| < \infty$. The space of functions of

bounded variation on $[0, 1]^n$ is a linear space, and the variation

is a norm for this space. Denote the space by bv^n .

If f is a real function defined on $[0, 1]^n$ and $\delta > 0$, define $f^\delta(x)$ for $x \in [0, 1]^n$ by

$$f^\delta(x) = f\left(\frac{x + \delta e}{1 + 2\delta}\right) - f\left(\frac{\delta e}{1 + 2\delta}\right),$$

where $e = (1, \dots, 1)$.

^{*}Cf. the proof of Lemma 7.24.

PROPOSITION 9.7. Let f be a continuous non-
decreasing real function on the unit cube $[0, 1]^n$ such
that for each i , the partial derivative $f_i = \partial f / \partial x_i$
exists^{*} and is continuous whenever $x_i > 0$. Then for
each $\delta > 0$, f^δ and f are in bv^n , and

$$\|f^\delta - f\| \rightarrow 0 \quad \text{as } \delta \rightarrow 0.$$

Proof. From the fact that f is nondecreasing it follows that f^δ also is, and hence at once that both are in bv^n .

For a given ϵ , let δ_1 correspond to $\epsilon/3n$ in accordance with Proposition 9.3; we may assume that $\delta_1 \leq 1$. For each i , the derivative f_i is uniformly continuous in $\{x \in [0, 1]^n : x_i \geq \delta_1/2\}$; therefore we may choose $\delta_2 > 0$ so that for all i and all x and y in the cube with $\|x - y\| < \delta_2$ and $x_i \geq \delta_1/2$, $y_i \geq \delta_1/2$, we have

$$(9.8) \quad |f_i(y) - f_i(x)| < \epsilon/6n.$$

Let β be a bound for $f_i(x)$ for all i and all x in the cube with $x_i \geq \delta_1/2$.

Now choose δ to satisfy the following conditions:

$$(9.9) \quad \delta \leq \epsilon/6\beta n$$

$$(9.10) \quad \delta \leq \delta_2$$

$$(9.11) \quad \delta \leq \delta_1/4.$$

^{*}When $x_i = 1$ the derivative is one-sided.

Let $g = f^\delta - f$. For an arbitrary i and x in the cube with $x_i \geq \delta_1/2$, and $\gamma \geq 0$ such that $x + \gamma e^i \in [0, 1]^n$, we have from the mean value theorem that

$$|g(x + \gamma e^i) - g(x)| = |g^i(x + \theta \gamma e^i)|_\gamma,$$

where $0 \leq \theta \leq 1$. Setting $y = x + \theta \gamma e^i$, and $z = (y + \delta e)/(1 + 2\delta)$, we obtain $y_i \geq x_i \geq \delta_1/2$. Further, if $y_i \leq \frac{1}{2}$, then $\delta \geq 2\delta y_i$, and hence

$$(9.12) \quad z_i = (y_i + \delta)/(1 + 2\delta) \geq y_i \geq \delta_1/2;$$

and if $y_i \geq \frac{1}{2}$, then

$$z_i \geq (\frac{1}{2} + \delta)/(1 + 2\delta) = \frac{1}{2} \geq \delta_1/2,$$

because $\delta_1 \leq 1$; so that (9.12) is in any case established. From (9.10) it follows that

$$\|z - y\| = \|e - 2y\| \frac{\delta}{1 + 2\delta} \leq \frac{\delta}{1 + 2\delta} \leq \delta \leq \delta_2.$$

From these observations and (9.8) and (9.9) it follows that

$$\begin{aligned} |g(x + \gamma e^i) - g(x)| &= |g^i(y)|_\gamma = \left| \frac{1}{1 + 2\delta} f_i\left(\frac{y + \delta e}{1 + 2\delta}\right) - f_i(y) \right|_\gamma \\ (9.13) \quad &= |(f_i(z) - f_i(y)) - \frac{2\delta}{1 + 2\delta} f_i(z)|_\gamma \\ &\leq (|f_i(z) - f_i(y)| + 2\delta\beta)_\gamma \leq \left(\frac{\epsilon}{6n} + \frac{\epsilon}{6n}\right)_\gamma \\ &= \frac{\epsilon}{3n} \gamma. \end{aligned}$$

Suppose now that

$$(9.14) \quad 0 = x^0 \leq \dots \leq x^k = e;$$

we wish to prove that

$$(9.15) \quad \sum_{j=1}^k |g(x^j) - g(x^{j-1})| < \epsilon.$$

Without loss of generality we may assume that

$$\|x^j - x^{j-1}\| < \delta_1/4$$

for all j ; for if this is not already the case, it can be made so by inserting additional points into the sequence (9.14), without decreasing the left side of (9.15); so that if (9.15) holds for the new sequence, it certainly holds for the original sequence.

For each i in $\{0, 1, \dots, n\}$ and each j in $\{1, \dots, k\}$, let

$$x^{ji} = (x_1^j, \dots, x_i^j, x_{i+1}^{j-1}, \dots, x_n^{j-1}).$$

Note that $x^{j0} = x^{j-1, n} = x^{j-1}$; and that x^{ji} and $x^{j, i-1}$ differ in the i 'th coordinate only. Then

$$g(x^j) - g(x^{j-1}) = g(x^{jn}) - g(x^{j0}) = \sum_{j=1}^n (g(x^{ji}) - g(x^{j-1, i})),$$

and hence

$$(9.16) \quad \sum_{j=1}^k |g(x^j) - g(x^{j-1})| \leq \sum_{i=1}^n \sum_{j=1}^k |g(x^{ji}) - g(x^{j-1, i})|.$$

Consider now an arbitrary but fixed i . We have

$$0 = x_i^0 \leq x_i^1 \leq \dots \leq x_i^k = 1.$$

Let $m = m(i)$ be such that

$$\delta_1/2 \leq x_i^m \leq \frac{3}{4}\delta_1 \leq \delta_1;$$

the existence of such an m follows from $\|x^j - x^{j-1}\| < \delta_1/4$. Since x^{ji} and $x^{j, i-1}$ differ in the i 'th coordinate only, and $x_i^{mi} = x_i^m \leq \delta_1$, it follows from the definition of δ_1 (i. e., from Proposition 9.3) that

$$\sum_{j=1}^m |f(x^{ji}) - f(x^{j, i-1})| < \epsilon/3n.$$

Furthermore, from $x_i^m \leq \frac{3}{4}\delta_1$ and (9.11) it follows that

$$(x^{mi} + \delta e)_i / (1 + 2\delta) \leq \delta_1,$$

so again using the definition of δ_1 , we deduce

$$\sum_{j=1}^m |f^\delta(x^{ji}) - f^\delta(x^{j, i-1})| < \epsilon/3n.$$

Hence

$$\sum_{j=1}^m |g(x^{ji}) - g(x^{j, i-1})| < 2\epsilon/3n.$$

Since $x_i^{m+1, i-1} = x_i^m \geq \delta_1/2$, we have from (9.13) that

$$\sum_{j=m+1}^k |g(x^{ji}) - g(x^{j, i-1})| \leq \frac{\epsilon}{3n} \sum_{j=m+1}^k (x_j^i - x_i^{j-1}) \leq \frac{\epsilon}{3n}.$$

Summing up, we obtain

$$\sum_{i=1}^n \sum_{j=1}^k |g(x^{ji}) - g(x^{j-1, i})| \leq \sum_{i=1}^n \epsilon/n \leq \epsilon,$$

and by applying (9.16), we complete the proof of Proposition 9.7.

PROPOSITION 9.17. Let f be a continuous non-
decreasing real function on the nonnegative orthant
of E^n , such that for each i , the partial derivative
 $\partial f / \partial x_i$ exists and is continuous whenever $x_i > 0$.
Let μ be an n -dimensional vector of nonnegative
measures. Then $f \cdot \mu \in \text{pNA}$.

Proof. Since the range R of μ is compact by Lyapunov's theorem [L], we may without loss of generality assume that $R \subset [0, 1]^n$ (otherwise replace $f(x)$ by $f(x/k)$ and μ by $k\mu$). For any function $g \in \text{bv}^n$, we then have $g \cdot \mu \in \text{BV}$, and $\|g \cdot \mu\| \leq \|g\|$. Since $f^\delta \cdot \mu - f \cdot \mu = (f^\delta - f) \cdot \mu$, it follows that $\|f^\delta \cdot \mu - f \cdot \mu\| \rightarrow 0$ as $\delta \rightarrow 0$. But $f^\delta \cdot \mu \in \text{pNA}$ because of Theorem B; hence also $f \cdot \mu \in \text{pNA}$, because pNA is closed. This completes the proof.

It is not difficult to extend the proof of Proposition 9.7 so that it holds when the condition $x_i > 0$ is replaced by $0 < x_i < 1$. In other words, we may prove the proposition for continuous non-decreasing functions defined on the unit cube whose partial derivatives exist whenever they are defined as two-sided derivatives. Proposition 9.17 therefore also holds for such functions, provided of course that the range of μ is in the unit cube. An appropriate analogue of Proposition 9.3 also holds.

More generally, let X be a closed convex subset of E^n containing the origin, and let f be a real continuous nondecreasing

function defined on X such that f is continuously differentiable on $\text{Rel Int } X$ (the relative interior of X). If x is in the relative boundary of X , and z is X -admissible, then f is said to be continuously differentiable at x in the direction z if it is continuously differentiable on $\{x\} \cup \text{Rel Int } X$ in the direction z .

Next, z is called tangent to X at x if $x + z$ is in the intersection of all the hyperplanes through x that support X . It seems reasonable to conjecture the following generalization of Proposition 9.17:

Assume that f is continuously differentiable on $\text{Rel Int } X$, and that for each x in the relative boundary of X and for each X -admissible z tangent to X at x , f is continuously differentiable at x in the direction z . Let μ be a vector of nonnegative measures whose range is included in X . Then $f \cdot \mu \in \text{pNA}$.

10. GENERALIZATIONS

Generalizations of the framework and the results of this paper are conceivable in a number of directions; the results are scattered, and many of the questions we will mention here have not been investigated at all. In some cases, the generalizations will lead us to rather pathological measure-theoretic considerations.

First, the definition of value may be generalized to non-symmetric subspaces of BV as follows:

DEFINITION 10.1. Let Q be a subspace of BV. A value on Q is a positive linear mapping φ from Q into FA satisfying the following conditions:

(10.2) If $v \in Q$ and $\Theta \in \mathcal{H}$ are such that $\Theta_* v \in Q$, then

$$\varphi(\Theta_* v) = \Theta_* \varphi v.$$

(10.3) $(\varphi v)(I) = v(I)$ for all $v \in Q$.

We will henceforth use the term "value" in the sense of this definition; no confusion can result, because when Q is symmetric, the definition coincides with that of Sec. 2.

The following generalization of Proposition 6.1 holds:

PROPOSITION 10.4. Proposition 6.1 holds as it stands if the word "symmetric" is omitted.

Proof. As in the proof of Proposition 6.1, assume that $(I, \mathcal{C}) = ([0, 1], \mathcal{B})$. If μ is Lebesgue measure λ , the proof

proceeds as in that proof. Otherwise, use Lemma 6.2 to obtain an automorphism Φ of the player space such that $\Phi_* \mu = \lambda$, which induces a positive linear one-one operator Φ_* of BV onto itself. Define a positive linear operator ϖ' from $\Phi_* Q$ into FA by

$$\varpi' = \Phi_* \varpi \Phi_*^{-1}.$$

We claim that ϖ' is a value on $\Phi_* Q$. Indeed, let Θ in \mathbb{C} and w in $\Phi_* Q$ be such that $\Theta_* w \in \Phi_* Q$. Then if $v = \Phi_*^{-1} w$, then $v \in Q$ and $(\Phi_*^{-1} \Theta \Phi)_* v = \Phi_*^{-1} \Theta_* \Phi_* \Phi_*^{-1} w \in Q$. Hence because ϖ is a value, we have

$$\begin{aligned} \varpi' \Theta_* w &= \Phi_* \varpi \Phi_*^{-1} \Theta_* \Phi_* \Phi_*^{-1} w = \Phi_* \varpi (\Phi_*^{-1} \Theta \Phi)_* v \\ &= \Phi_* (\Phi_*^{-1} \Theta \Phi)_* \varpi v = \Phi_* \Phi_*^{-1} \Theta_* \Phi_* \varpi \Phi_*^{-1} w = \Theta_* \varpi' w \end{aligned}$$

proving (10.2); the proof of (10.3) is trivial.

Let $v = f \cdot \mu$. Then $v \in Q$ and so $\Phi_* v \in \Phi_* Q$. Now

$$(\Phi_* v)(S) = v(\Phi S) = f(\mu(\Phi S)) = f(\lambda(S)) = (f \cdot \lambda)(S),$$

so $f \cdot \lambda \in \Phi_* Q$. Since ϖ' is a value on $\Phi_* Q$, we may deduce from what we have already shown that $\varpi'(\Phi_* v) = \varpi'(f \cdot \lambda) = \lambda$. Hence $\varpi v = \Phi_*^{-1} \varpi' \Phi_* v = \Phi_*^{-1} \lambda$; so

$$(\varpi v)(S) = (\Phi_*^{-1} \lambda)(S) = \lambda(\Phi^{-1} S) = \mu(\Phi \Phi^{-1} S) = \mu(S),$$

and the proof of Proposition 10.4 is complete.

Proposition 6.4 is more difficult to generalize to the non-symmetric case. If one restricts oneself to subspaces of $bv' NA$,

there is of course no difficulty; the difficulty occurs if one admits functions f_α that may have jumps at 0 or at 1. This situation remains obscure at the present.

Another direction in which the results can be generalized concerns the underlying space. We can drop assumption (2.1), i. e., allow the underlying space to be an arbitrary measurable space, with the proviso that the all-player set I is measurable. The definition of value applies to this situation without any change; so does all of Sec. 4, and Proposition 6.4. From this it follows that the existence statements—but not the uniqueness statements!—for values in Theorems A and B also apply to this more general situation. So do the second part of Theorem B, Theorem C, and the results (8.25), (8.26) on the geometry of $bv'NA$.

The difficulties begin with Proposition 6.1, and they will lead us to some rather pathological aspects of measure theory. To understand these difficulties, let us reexamine the argument used in Sec. 3 as an intuitive support for $\phi(f \cdot \mu) = \mu$: "In the game $f \cdot \mu$, the payoff to a coalition depends only on its μ -measure. There would therefore seem to be no reason to 'discriminate' between coalitions having equal μ -measure, given assumption (2.1), and it seems natural to conjecture that they should get the same value. Because of the non-atomicity and the normalization condition (2.3), this implies that the value equals the μ -measure."

The first portion of this argument is less than convincing in the context of an arbitrary measure space, because it ignores the fact that sets having the same μ -measure may yet be distinguishable because of the underlying structure of the space I of players. The simplest example of this phenomenon is that in which some individual points have positive μ -measure (see [M-S]). This violates the non-atomicity, but the first part of the argument does not really depend on non-atomicity; this is used only in the last part.

Another example, in which the measure is non-atomic, is as follows:

Example 10.5. Let $I = I_1 \cup I_2$, where $I_1 \cap I_2 = \emptyset$, I_1 is the cartesian product of denumerably many copies of the discrete space $\{0, 1\}$, and I_2 is the cartesian product of continuum many copies of $\{0, 1\}$. The σ -field \mathcal{C} on I is that generated by the product σ -fields on I_1 and I_2 respectively. On $\{0, 1\}$ we define a measure ν by letting $\nu(0) = \nu(1) = \frac{1}{2}$; μ_1 is then the product measure on I_1 , μ_2 the product measure on I_2 , and we define μ on I by $\mu(S) = \frac{1}{2}\mu_1(S \cap I_1) + \frac{1}{2}\mu_2(S \cap I_2)$. We can now define a set function $f \cdot \mu$ by

$$f(x) = \begin{cases} 0 & \text{for } x \leq \frac{1}{2} \\ 1 & \text{for } x > \frac{1}{2} \end{cases}$$

In this example, the measure is non-atomic, and I_1 and I_2 have the same measure. But I_1 has a smaller cardinality, and it is therefore conceivable that we might wish to assign a larger value to I_1 than to I_2 , or at any rate a different value. Indeed, we may think of this set function as a voting game, in which $\mu(S)$ is the voting strength or "weight" of coalition S , and a majority "wins". Since $\mu(I_1) = \mu(I_2)$ but I_1 has smaller cardinality, we may conclude that the "weight per player" in I_2 is infinitesimal compared with that in I_1 . So we might expect that the total value of I_1 would be different from that of I_2 , for roughly the same reason that in finite voting games, players of different individual weights often get more or less than a proportional share of the value. Of course this is not necessarily so; we are only pointing out that it is not so clear that sets of equal measure must get the same value, even if the payoff depends only on the measure and the measure is non-atomic.

For the above reasons, it seems reasonable to believe that

(10.6) for the player space of Example 10.5, there is a closed symmetric reproducing subspace of BV on which there is a value m that does not obey Proposition 6.1.

This is indeed the case. The crux of the proof is that each automorphism θ of (I, \mathcal{C}) leaves both I_1 and I_2 fixed. To see this we first prove

LEMMA 10.7. Every nonempty measurable subset of I_2 has 2^c elements, where c is the cardinality of the continuum.

Proof. Let $I_2 = \prod_{\beta} \Omega_{\beta}$, where each Ω_{β} is a copy of $\{0, 1\}$, and β ranges over an index set C of cardinality c . We claim that each measurable subset S of I_2 is of the form $S' \times \prod_{\beta \in C \setminus D} \Omega_{\beta}$, where D is a denumerable subset of C , and $S' \subset \prod_{\beta \in D} \Omega_{\beta}$. This is certainly true of the sets that generate the σ -field of I_2 , for which D contains only one element and $S' = \{0\}$ or $S' = \{1\}$. Furthermore, if our claim holds for a given S , then it also holds for its complement $I_2 \setminus S$; and if it holds for a sequence S_1, S_2, \dots , then it also holds for its union $\bigcup_i S_i$. A standard argument then yields the truth of our claim, and the lemma follows.

Now if Θ is an automorphism, then $I_2 \cap \Theta(I_1)$ is measurable; it must be of cardinality $\leq c$ because it is included in $\Theta(I_1)$, and either empty or of cardinality $\geq 2^c$ because it is a measurable subset of I_2 ; so it is empty, i. e., $\Theta(I_1) \subset I_1$. Similarly $\Theta^{-1}(I_1) \subset I_1$, i. e., $I_1 \subset \Theta(I_1)$; so

$$I_1 = \Theta(I_1)$$

and hence also $I_2 = \Theta(I_2)$.

Let us call a measure μ on I balanced if $\mu(I_1) = \mu(I_2)$; let BNA be the space of all balanced non-atomic measures on I .

What we have shown implies that BNA is symmetric. Now define

φ by

$$(10.8) \quad (\varphi\mu)(S) = 2\mu(S \cap I_1);$$

that is, give all the value to I_1 . Clearly φ does not satisfy

Proposition 6.1. On the other hand

$$(10.9) \quad (\varphi_*\mu)(S) = (\varphi\mu)(\varphi S) = 2\mu((\varphi S) \cap I_1) \\ = 2\mu(\varphi(S \cap I_1)) = 2(\varphi_*\mu)(S \cap I_1) = (\varphi\varphi_*\mu)(S),$$

so that φ does satisfy (2.2). It is easily seen that it satisfies (2.3)

and is positive, so it is a value, and the proof of (10.6) is complete.

Let us recall that a measurable space is called separable if its σ -field has a denumerable generating subset. Does (10.6) depend essentially on the nonseparability of the underlying space, by all odds a rather pathological feature? To answer this question, note that the value defined by (10.8) is not the only one that violates Proposition 6.1; alternatively, φ could have been defined by $\varphi(\mu) = 2\mu(S \cap I_2)$, i. e., we could have assigned all the value to the "smaller" players of I_2 rather than the "larger" players of I_1 . This suggests that it is not the size of the players that "makes" the example, but rather the extreme inhomogeneity, the fact that there are no transformations that can "mix" I_1 and I_2 . Can a similar inhomogeneity occur even when the player space is separable

The answer is yes. To construct the example, we need the following lemma:

LEMMA 10.10. There is a subset I_2 of $(1, 2]$ that includes no nondenumerable Borel subset of $(1, 2]$ but intersects every such set.

Proof. There are continuum many nondenumerable Borel subsets of $(1, 2]$; using the continuum hypothesis, denote them by B_α , where α runs over all ordinal numbers less than \aleph_1 . Similarly, let the points of $(1, 2]$ be x_α , where α runs over the same set of ordinal numbers. Define two sets $\{y_\alpha\}$ and $\{z_\alpha\}$, where α again runs over all ordinals $< \aleph_1$, as follows: y_1 is the first point in B_1 (i. e., the x_α with smallest α), and z_1 is the second point in B_1 . If y_β and z_β have been defined for all ordinals less than a given denumerable ordinal α , define y_α to be the first point in B_α that is larger than all the points y_β and z_β already defined, and z_α to be the second such point; they exist because B_α is nondenumerable and only denumerably many points have already been defined. The proof is completed by setting $I_2 = \{y_\alpha\}_{\alpha < \aleph_1}$.

Example 10.11. Let I_1 be the unit interval $[0, 1]$, I_2 the subset of $(1, 2]$ defined in Lemma 10.10, and $I = I_1 \cup I_2$. Define \mathcal{C} to be the set of subsets of I of the form $I \cap U$, where U is a Borel subset of $[0, 2]$. Then

(10.12) there is a closed symmetric subspace Q of BV
on which there is a value φ that does not
satisfy Proposition 6.1.

The proof of (10.12) will parallel that of (10.6). As in that case, it depends on a certain inhomogeneity of the underlying space, though here it is not quite as clear cut as there.

LEMMA 10.13. If Θ is an automorphism of
the player space (I, \mathcal{C}) , then both $\Theta I_1 \setminus I_1$ and
 $\Theta^{-1} I_1 \setminus I_1$ are denumerable.

Proof. The concept of a subspace of a given measurable space plays a central role in the proof. Let (J, \mathcal{A}) be a measurable space, and let S be a (not necessarily measurable) subset of J . If we define \mathcal{E} to be the set of all sets of the form $S \cap T$, where $T \in \mathcal{A}$, then \mathcal{E} is called the subspace σ -field, and (S, \mathcal{E}) is a subspace of (J, \mathcal{A}) ; sometimes we refer to this subspace simply as S , and it is to be understood that the σ -field to be attached to S is the subspace σ -field. Note that if $J \supset S \supset T$, then T considered as a subspace of S is the same as T considered as a subspace of J .

The measurable spaces involved in this proof are $[0, 2]$ with its Borel subsets, and the subspaces I , I_1 , and ΘI_1 of $[0, 2]$.

From the fact that ΘI_1 may also be considered a subspace of I ,

it follows that Θ is an isomorphism from I_1 onto ΘI_1 . Let ϕ be the inclusion mapping from ΘI_1 into $[0, 2]$. It then follows that $\phi\Theta$ is a measurable function from I_1 into $[0, 2]$, whose range is ΘI_1 .

We now use a theorem of Mackey [M, Theorem 3.2, p. 139], which states that if ψ is a one-one measurable function from $[0, 1]$ into the real line, both endowed with the Borel σ -field, then the range of ψ is a Borel set. Applying this for $\psi = \phi\Theta$, we deduce that ΘI_1 is a Borel set. Hence $\Theta I_1 \setminus I_1$ is a Borel set, but since it is included in I_2 , it cannot be nondenumerable; this proves that $\Theta I_1 \setminus I_1$ is denumerable. If we substitute Θ^{-1} for Θ , we get the second half of the lemma, and this completes its proof.

As in the previous proof, let BNA be the space of all balanced non-atomic measures μ on I , where "balanced" means, as before, that $\mu(I_1) = \mu(I_2)$; we show that BNA is nonempty. Indeed, let μ be any non-atomic measure on $[0, 2]$. Now every coalition S in our player space is of the form $T \cap I$, where T is Borel; then because I meets every nondenumerable Borel set in $[0, 2]$, we may define a measure μ' on I by

$$\mu'(S) = \mu(T),$$

and this measure is balanced. BNA is a closed subspace of BV, and because the measures are non-atomic, Lemma (10.13) implies that it is symmetric. The remainder of the proof of (10.12) is exactly as in that of (10.6).

An esthetic flaw in these two counterexamples is that they are based on set functions that are measures, whose values are different measures. This certainly does not contradict our definitions; however, considerations of "dummies" or "strategic equivalence" [N-M, p. 245] might make it not unreasonable to add to the definition of value a "projection" axiom, namely that

$$(10.14) \quad \text{if } v \in FA, \text{ then } \varphi v = v.$$

The question arises whether the previous counterexamples can be modified so that the values satisfy (10.14), but still do not satisfy Proposition 6.1.

The answer is yes. In either of the two examples, let $pBNA$ be the space spanned by positive integer powers of members of BNA , and $s'BNA$ the space spanned by functions of the form $f \cdot \mu$, where $\mu \in BNA$ is a probability measure and $f \in bv'$ is singular. The spaces BNA , $pBNA$, and $s'BNA$ are all closed and symmetric. Since $BNA \subset NA$, there is a value ψ on $bv'BNA = pBNA + s'BNA$ satisfying Proposition 6.1. Define a continuous linear mapping θ from $s'BNA$ to M by

$$\theta = \varphi \psi,$$

where φ is defined by (10.8); clearly θ does not satisfy Proposition 6.1. On the other hand, by using (10.9) and the fact that ψ is a value, we obtain

$$\theta_* \theta = \theta_* \psi = \psi \theta_* = \psi \theta_* = \theta_*$$

for any $\theta \in \mathcal{H}$, so that θ satisfies (2.2). It is easily seen that it satisfies (2.3) and is positive, so it is a value. Finally, from (8.26) it follows that $s'BNA$ contains no nontrivial measures, so (10.14) is vacuously satisfied.

Furthermore, because $pBNA \cap s'BNA = \{0\}$, from (8.26), every member of $bv'BNA = pBNA + s'BNA$ has a unique representation in the form $v + w$ where $v \in pBNA$ and $w \in s'BNA$. Hence we may extend the value θ from $s'BNA$ to $pBNA + s'BNA$ by

$$\theta(v + w) = \psi v + \theta w.$$

Then θ does not satisfy Proposition 6.1, is a value, and satisfies (10.14) nonvacuously.

Appendix A

PROOF OF LEMMA 8.5

Assume w.l.o.g. that g is not constant. Let

$$[\alpha, \beta] = g(E^1) = [\min g, \max g],$$

and let $(a, b) = g^{-1}((\alpha, \beta))$; then $[a, b]$ is a support of g , and in fact the smallest support. Define a function \hat{g} in bv^* by

$$\hat{g}(\tau) = \begin{cases} a & , \quad \text{when } \tau < \alpha; \\ \max g^{-1}(\tau), & \text{when } \alpha \leq \tau < \beta; \\ b & , \quad \text{when } \beta \leq \tau. \end{cases}$$

Then \hat{g} is right continuous, is increasing in $[\alpha, \beta]$, and is continuous at β . Furthermore, for $\tau \in [\alpha, \beta]$ we have

$$g^{-1}(-\infty, \tau] = (-\infty, \hat{g}(\tau)],$$

and for $\tau \in [\alpha, \beta]$ we have

$$g(\hat{g}(\tau)) = \tau.$$

Let $h = f \cdot \hat{g}$, and let $t \in (a, b)$ be such that $h'(g(t))$ exists, and such that g is univalent at t , i.e., takes the value $g(t)$ at t only. Then we claim that

$$(A.1) \quad f_{(g)}(t) \text{ exists and } = h'(g(t)).$$

To prove this, set $\tau = g(t)$, and note that $\alpha < \tau < \beta$.

Furthermore since $g(\hat{g}(g(t))) = g(t)$, it follows from the univalency that

$$\hat{g}(\tau) = \hat{g}(g(t)) = t.$$

From the definition of h' , it follows that

$$\begin{aligned} h'(\tau) &= \lim_{\sigma \rightarrow \tau} \frac{f(\hat{g}(\sigma)) - f(\hat{g}(\tau))}{\sigma - \tau} \\ &= \lim_{\sigma \rightarrow \tau} \frac{f(\hat{g}(\sigma)) - f(t)}{\sigma - g(t)}. \end{aligned}$$

Hence for each $\epsilon > 0$, there is a $\delta > 0$ such that

$$(A.2) \quad \sigma \neq \tau \text{ and } |\sigma - \tau| < \delta \Rightarrow \left| \frac{f(\hat{g}(\sigma)) - f(t)}{\sigma - g(t)} - h'(\tau) \right| < \epsilon.$$

Let Δ denote the interval $(\tau - \delta, \tau + \delta)$, and choose δ sufficiently small so that $\Delta \subset (\alpha, \beta)$. Then because of the continuity of g , the inverse image $g^{-1}(\Delta)$ is an open interval containing t . If $s \in g^{-1}(\Delta)$ and $s \neq t$, then $g(s) \in \Delta$ and $g(s) \neq g(t)$, and we may approach $g(s)$ from above and from below by points ρ in Δ . If $\rho > g(s)$, then $g(\hat{g}(\rho)) = \rho > g(s)$, so because g is monotonic,

$$\hat{g}(\rho) > s;$$

similarly if $\rho < g(s)$, then $g(\hat{g}(\rho)) = \rho < g(s)$, so

$$\hat{g}(\rho) < s.$$

Let

$$s^+ = \lim_{\rho \rightarrow g(s)+} \hat{g}(\rho)$$

$$s^- = \lim_{\rho \rightarrow g(s)-} \hat{g}(\rho).$$

From the previous remarks it follows that

$$(A.3) \quad s^- \leq s \leq s^+.$$

Since $\rho \in \Delta$, we have $|\rho - \tau| < \delta$; so applying (A.2), we deduce

$$\left| \frac{f(\hat{g}(\rho)) - f(t)}{\rho - g(t)} - h'(\tau) \right| < \epsilon.$$

Letting $\rho \rightarrow g(s)+$ and $\rho \rightarrow g(s)-$ respectively, we obtain

$$\left| \frac{f(s^+) - f(t)}{g(s) - g(t)} - h'(\tau) \right| \leq \epsilon$$

and

$$\left| \frac{f(s^-) - f(t)}{g(s) - g(t)} - h'(\tau) \right| \leq \epsilon.$$

Hence from (A.3) and the monotonicity of f it follows that

$$(A.4) \quad \left| \frac{f(s) - f(t)}{g(s) - g(t)} - h'(\tau) \right| \leq \epsilon.$$

So if we let δ' be such that $|s - t| < \delta' \Rightarrow s \in g^{-1}(\Delta)$, then we have shown that for each ϵ there is a δ' such that

$$s \neq t \text{ and } |s - t| < \delta' \Rightarrow (A.4).$$

This proves (A.1).

Let Γ be the set of points in $[a, b]$ that are taken on more than once. Then for each τ in Γ , $g^{-1}(\tau)$ is a non-denumerable interval; hence Γ is denumerable, and in particular $\lambda(\Gamma) = 0$, where λ is Lebesgue measure.

Let Ξ be the set of points τ in $[a, b]$ such that $h'(\tau)$ does not exist at τ . Then also $\lambda(\Xi) = 0$, and hence $\lambda(\Gamma \cup \Xi) = 0$. From (A.1) it follows that $f_{(g)}(t)$ exists except possibly when $t \in g^{-1}(\Gamma \cup \Xi)$. On the other hand, because g is continuous and monotonic, and because in particular it takes only denumerably many values more than once, it follows that $\lambda \circ g$ is a measure on the Borel sets of R , and hence

$$\nu_g = \lambda \circ g;$$

for later reference we note that for the same reasons,

$$\nu_f = \lambda \circ f.$$

Therefore if we let $\tilde{\lambda}$ be the restriction of λ to $[\alpha, \beta]$, i.e., $\tilde{\lambda}(S) = \lambda(S \cap [\alpha, \beta])$, then

$$\nu_g \circ g^{-1} = \tilde{\lambda}.$$

Hence

$$\nu_g(g^{-1}(\Gamma \cap \Xi)) = \tilde{\lambda}(\Gamma \cap \Xi) = \lambda(\Gamma \cap \Xi) = 0,$$

so $f_{(g)}(t)$ exists a.e. w.r.t. ν_g .

Since also $\nu_g(g^{-1}(\Gamma)) = \lambda(\Gamma) = 0$, it follows that $g(s)$ is taken on only once for almost all s (w.r.t. ν_g), and hence that

$$(A.5) \quad \hat{g}(g(s)) = s \text{ a.e. w.r.t. } \nu_g.$$

Next, we establish

$$(A.6) \quad f \ll g \Rightarrow h \text{ is absolutely continuous.}$$

$$(A.7) \quad f \perp g \Rightarrow h \text{ is singular.}$$

To prove (A.6), we deduce from $f \ll g$ and the Radon-Nikodym theorem that

$$f(t) = f(0) + \int_0^t \varphi(s) dg(s),$$

where φ is Borel-measurable. Applying this formula for $t = \hat{g}(\tau)$, when $\tau \in [\alpha, \beta)$, changing variables [H₁, p. 163], and recalling $\nu_g \cdot g^{-1} = \tilde{\lambda}$, we obtain

$$\begin{aligned} h(\tau) - h(\alpha) &= f(\hat{g}(\tau)) - f(\hat{g}(\alpha)) \\ &= \int_a^{\hat{g}(\tau)} \varphi(s) dg(s) = \int_{-\infty}^{\hat{g}(\tau)} \varphi(s) dg(s) \\ &= \int_{g^{-1}(-\infty, \tau]} \varphi(\hat{g}(g(s))) d\nu_g(s) \\ &= \int_{(-\infty, \tau]} \varphi(\hat{g}(\sigma)) d\nu_{g^{-1}}(\sigma) \\ &= \int_{-\infty}^{\tau} \varphi(\hat{g}(\sigma)) d\tilde{\lambda}(\sigma) \\ &= \int_{\alpha}^{\tau} \varphi(\hat{g}(\sigma)) d\sigma. \end{aligned}$$

Hence h is absolutely continuous on $[\alpha, \beta)$. Since \hat{g} is continuous at β , so is h ; so h is absolutely continuous on $[\alpha, \beta]$, which is a support of h . Hence h is absolutely continuous, and (A.6) is proved.

To prove (A.7), let $U \subset [a, b]$ be such that $\nu_f([a, b] \setminus U) = 0$ and $\nu_g(U) = 0$. Since $\nu_g = \lambda \circ g$, it follows that

$$\lambda(g(U)) = 0.$$

Next, for $\alpha \leq \tau < \beta$ we have

$$\begin{aligned} \nu_h(-\infty, \tau] &= \nu_h[\alpha, \tau] = h^c(\tau) - h^c(\alpha) \leq h(\tau) - h(\alpha) \\ &= f(\hat{g}(\tau)) - f(a) = \lambda f[a, \hat{g}(\tau)] \\ &= \nu_f([a, b] \cap g^{-1}(-\infty, \tau]). \end{aligned}$$

Since both ν_h and $\nu_f([a, b] \cap g^{-1}(S))$ are non-negative measures, it follows that

$$\nu_h(S) \leq \nu_f([a, b] \cap g^{-1}(S))$$

for all $S \subset (-\infty, \beta)$; and since $\nu_h[\beta, \infty) = 0$, this holds for all $S \subset E^1$.

Now $g^{-1}(g(U)) \supset U$, so

$$g^{-1}(E^1 \setminus g(U)) = g^{-1}(E^1) \setminus g^{-1}g(U) \subset E^1 \setminus U.$$

Hence

$$\nu_h(E^1 \setminus g(U)) \leq \nu_f([a, b] \cap (E^1 \setminus U)) = \nu_f([a, b] \setminus U) = 0.$$

Hence all the increase of the function h takes place in the set $g(U)$, which is of Lebesgue measure 0; and this establishes (A.7).

Returning to the case of an arbitrary (continuous and nondecreasing) f , let $h_1 = f^{ac} \cdot \hat{g}$, $h_2 = f^\perp \cdot \hat{g}$; then $h = h_1 + h_2$, and from (A.6) and (A.7) it follows that $h_1 \ll \text{identity}$, $h_2 \perp \text{identity}$. Hence $h = h_1 + h_2$ is the decomposition of h w.r.t. the identity, and hence by (8.4) and (A.1), we have

$$h_1(\tau) - h_1(\alpha) = \int_{\alpha}^{\tau} h_1'(\sigma) d\sigma = \int_{\alpha}^{\tau} f_{(g)}(\hat{g}(\sigma)) d\sigma.$$

Therefore for $t = \hat{g}(g(t))$ (which implies $t \in [a, b]$), a change of variables yields

$$\begin{aligned} \text{(A.8)} \quad f^{ac}(t) &= f^{ac}(\hat{g}(g(t))) = h_1(g(t)) \\ &= h_1(\alpha) + \int_{\alpha}^{g(t)} f_{(g)}(\hat{g}(\sigma)) d\sigma \\ &= h_1(a) + \int_{[\alpha, g(t)]} f_{(g)}(\hat{g}(\sigma)) dv_g^{-1}(s) \\ &= \bar{f}^{ac}(a) + \int_{(-\infty, \hat{g}(g(t))]} f_{(g)}(\hat{g}(g(s))) dg(\sigma) \\ &= f^{ac}(a) + \int_a^t f_{(g)}(s) dg(s), \end{aligned}$$

where the last equality follows from (A.5).

If $t < \hat{g}(g(t))$, let $t^* = \hat{g}(g(t))$; certainly $\hat{g}(g(t^*)) = t^*$. Since g does not change in the interval $[t, t^*]$, it follows that $v_g[t, t^*] = 0$; hence from absolute continuity it follows that

$$f^{ac}(t) - f^{ac}(t^*) = v_{f^{ac}}[t, t^*] = 0.$$

But from $v_g[t, t^*] = 0$ it also follows that

$$\int_0^t f_{(g)}(s) dg(s) = \int_0^{t^*} f_{(g)}(s) dg(s),$$

and hence from (A.8) it follows that

$$(A.9) \quad f^{ac}(t) = f^{ac}(a) + \int_a^t f_{(g)}(s) dg(s).$$

The case $t > \hat{g}(g(t))$ can occur only if $t > a$, and (A.9) is easily verified in this case as well. So (A.9) always holds. The integrability of $f_{(g)}$ w.r.t. v_g follows easily, and the proof of Lemma 8.5 is complete.

Appendix B

ϵ -MONOTONICITY

For motivation of this Appendix, see the end of Sec. 4.

For $v \in BV$, define the downward variation of v to be

$$\sup \sum_i \max \{ (v(S_i) - v(S_{i+1})), 0 \},$$

where the sup is taken over all chains

$$\emptyset = S_0 \subset \dots \subset S_k = I.$$

v is said to be ϵ -monotonic if its downward variation is $\leq \epsilon$. Note that a set function is monotonic if and only if it is 0-monotonic.

The chief result we wish to prove here is

PROPOSITION B.1. Let Q be a subspace of BV with
 $NA^+ \subset Q \subset AC$; let $\epsilon > 0$; and let Q^ϵ denote the set of
 ϵ -monotonic elements of Q . Then

$$Q = Q^\epsilon - Q^\epsilon$$

(in fact, $Q = Q^\epsilon - Q^+$.)

For the proof we need the following.

LEMMA B.2. For any $v \in AC$ and $\epsilon > 0$, there exists
 $\mu \in NA^+$ and a real number $c > 0$ such that $v + c\mu$ is
 ϵ -monotonic.* Moreover, μ (but not c) may be chosen
independently of ϵ .

* That ϵ cannot be taken to be 0 can be seen by taking $v = \sqrt{\lambda}$, where λ is Lebesgue measure on $([0, 1], B)$.

Proof. By definition of AC, there is a $\mu \in NA^+$ such that for all $\epsilon > 0$ there is a $\delta > 0$ such that for any subchain Λ of any chain Ω

$$(B. 3) \quad \|\mu\|_{\Lambda} \leq \delta \Rightarrow \|v\|_{\Lambda} \leq \epsilon.$$

For any $S \subset T$ with $v(S) \neq v(T)$, define the ratio

$$\rho(S, T) = \frac{|v(T) - v(S)|}{\mu(T \setminus S)};$$

$\rho(S, T)$ may be infinite, and is always positive. Let Λ' denote the set of links $(S, T) \in \Lambda$ such that $\rho(S, T) > \frac{\epsilon}{\delta}$. We claim that

$$(B. 4) \quad \|\mu\|_{\Lambda'} < \delta.$$

Suppose not. Let $\eta > 0$ satisfy $\eta < \delta$, and interpolate sets R_j :

$$S = R_0 \subset R_1 \subset \dots \subset R_k = T$$

between the S, T of each link in Λ' in such a way that $\mu(R_j \setminus R_{j-1}) \leq \eta$, $j = 1, \dots, k$. Let Λ^* denote the set of all links $\{R_{j-1}, R_j\}$ obtained in this way; clearly Λ^* is a subchain of some chain Ω^* that "refines" Ω . Moreover

$$(B. 5) \quad \|v\|_{\Lambda^*} \geq \|v\|_{\Lambda'}, \text{ and } \|\mu\|_{\Lambda^*} = \|\mu\|_{\Lambda'}.$$

Now arrange the links of Λ^* according to their ρ values.

Choosing those of highest ρ value, we may, since (B. 4) is false, select enough of them to make a subchain $\Lambda^{*'} with$

$$\delta - \eta \leq \|\mu\|_{\Lambda^{*'}} \leq \delta.$$

Then by definition of δ ,

$$\|v\|_{\Lambda^*} \leq \epsilon.$$

Hence

$$(B.6) \quad \frac{\epsilon}{\delta - \eta} \geq \frac{\|v\|_{\Lambda^*}}{\|\mu\|_{\Lambda^*}}.$$

Since we used the highest ρ -values in Λ^* , we have

$$(B.7) \quad \frac{\|v\|_{\Lambda^*}}{\|\mu\|_{\Lambda^*}} \geq \frac{\|v\|_{\Lambda^*}}{\|\mu\|_{\Lambda^*}}.$$

By (B.5), we have

$$(B.8) \quad \frac{\|v\|_{\Lambda^*}}{\|\mu\|_{\Lambda^*}} \geq \frac{\|v\|_{\Lambda^*}}{\|\mu\|_{\Lambda^*}}.$$

If ρ_0 denotes the smallest ρ -value used in Λ^* , then we have

$$(B.9) \quad \frac{\|v\|_{\Lambda^*}}{\|\mu\|_{\Lambda^*}} \geq \rho_0 > \frac{\epsilon}{\delta}.$$

Now (B.6), (B.7), (B.8), (B.9) yield

$$\frac{\epsilon}{\delta - \eta} \geq \rho_0 > \frac{\epsilon}{\delta},$$

and as $\eta \rightarrow 0$ we obtain a contradiction. This proves (B.4). From

(B.3) and (B.4) we obtain

$$(B.10) \quad \|v\|_{\Lambda^*} \leq \epsilon.$$

Now define

$$w = v + \frac{\epsilon}{\delta} \mu.$$

For $(S, T) \in \Lambda \setminus \Lambda'$, we have

$$\begin{aligned} w(T) - w(S) &= v(T) - v(S) + \frac{\epsilon}{\delta} \mu(T \setminus S) \\ &\geq -|v(T) - v(S)| + \frac{\epsilon}{\delta} \mu(T \setminus S) \\ &= (-\rho(S, T) + \frac{\epsilon}{\delta}) \mu(T \setminus S) \\ &\geq 0. \end{aligned}$$

Hence

$$\begin{aligned} \text{(B.11)} \quad \sum_{(S, T) \in \Lambda} |\min(0, w(T) - w(S))| &= \sum_{\Lambda'} |\min(0, v(T) - v(S) + \frac{\epsilon}{\delta} \mu(T \setminus S))| \\ &\leq \sum_{\Lambda'} |v(T) - v(S)| \\ &= \|v\|_{\Lambda'} \leq \epsilon \quad \text{by (B.10).} \end{aligned}$$

On the other hand, the downward variation of w is the supremum of the left side of (B.11), over all subchains Λ ; hence it is $\leq \epsilon$. This completes the proof of the lemma.

Proof of Proposition B. 1. Let $v \in Q$. Choose μ, c by the lemma so that $w = v + c\mu$ is ϵ -monotonic. Then $w \in Q^\epsilon$ and $c\mu \in Q^+ \subset Q^\epsilon$, and $v = w - c\mu$. This completes the proof of the proposition.

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